MACKAY'S

# ELEMENTS OF EUCLID.

BOOKS I to III.

Authorized by the Department of
Education.

PRICE 50 CENTS.

TORONTO:
ROSE PUBLISHING COMPANY.

DONATED BY

W. Kinrade

BY

H Konrade

Ellen Cherry
Perm
Ont.
S. S. No. 8
Mulmur.
1893.

Digitized by the Internet Archive
in 2009 with funding from
Ontario Council of University Libraries

http://www.archive.org/details/b1t3elementsofeuwest00mack

# THE ELEMENTS

OF

# EUCLID

BOOKS I. TO III.

*WITH*

*DEDUCTIONS, APPENDICES, AND HISTORICAL NOTES*

BY

JOHN STURGEON MACKAY, M.A., F.R.S.E.,

MATHEMATICAL MASTER IN THE EDINBURGH ACADEMY.

TORONTO:
ROSE PUBLISHING COMPANY
LONDON AND EDINBURGH: W. & R. CHAMBERS.

Entered according to Act of the Parliament of Canada, in the yea
one thousand eight hundred and eighty-seven, by W. & R.
HAMBERS, at the Department of Agriculture.

PRINTED BY
HUNTER, ROSE & CO.,
TORONTO.

# PREFACE.

In this text-book, compiled at the request of the publishers, a rigid adherence to Robert Simson's well-known editions of Euclid's *Elements* has not been observed; but no change has been made on Euclid's sequence of propositions, and comparatively little on his modes of proof. Here and there useful corollaries and converses have been inserted, and a few of Simson's additions have been omitted. Intimation of such insertions and omissions has been given, when it was deemed necessary, in the proper place. Several changes, mostly, however, of arrangement, have been made on the definitions.

By a slight alteration of the lettering or the construction of the figure, an attempt has been made throughout, and particularly in the Second Book, to draw the attention of the reader to the analogy which exists between certain pairs of propositions. By Euclid this analogy is well-nigh ignored.

In the naming of both congruent and similar figures, care has been taken to write the letters which denote corresponding points in a corresponding order. This is a matter of minor importance, but it does not deserve to be neglected, as is too often the case.

The deductions or exercises appended to the various propositions ('riders,' as they are sometimes termed) have been intentionally made easy and, in the First Book, numerous. It is hoped that beginners, who have little confidence in their own reasoning power, will thereby be encouraged to do more than merely learn the text of Euclid. It is hoped also that sufficient provision has been made for all classes of beginners, seeing that the questions, deductions, and corollaries to be proved number considerably over fifteen hundred. It should be stated that when a deduction is repeated once or oftener, in the same words, a different mode of proof is expected in each case.

In the appendices, much curtailed from considerations of space, a few of the more useful and interesting theorems of elementary geometry have been given. It has not been thought expedient to introduce the signs + and −, to indicate opposite directions of measurement. The important advantages which result from this use of these signs are readily apprehended by readers who advance beyond the 'elements,' and it is only of the 'elements' that the present manual treats.

The historical notes, which are not specially intended for beginners, may save time and trouble to any one who wishes to investigate more fully certain of the questions which occur throughout the work. It would perhaps be well if such notes were more frequently to be found in mathematical text-books: the names of those who have extended the boundaries, or successfully cultivated any part of the domain, of science should not be unknown to those who inherit the results of their labour.

Though the utmost pains have been taken by all concerned in the production of this volume to make it accurate and workmanlike, a few errors may have escaped notice. Corrections of these will be gratefully received.

The editor desires to express his thanks to Mr J. R. PAIRMAN for the excellence of the diagrams, and to Mr DAVID TRAILL, M.A., B.Sc., and Mr A. Y. FRASER, M.A., for valuable hints while the work was going through the press.

EDINBURGH ACADEMY,
*April* 1884.

# CONTENTS.

## BOOK I.

### *ANGLES, TRIANGLES, PARALLELS, PARALLELOGRAMS—*

PAGE

DEFINITIONS........ 1
POSTULATES........ 11
AXIOMS........ 11
QUESTIONS ON THE DEFINITIONS, POSTULATES, AXIOMS........ 13
EXPLANATION OF TERMS........ 18
SYMBOLS AND ABBREVIATIONS........ 19
PROPOSITIONS 1–48 WITH RIDERS........ 21
PROPOSITION A........ 61

APPENDIX I.

PROPOSITIONS 1–5........ 98
LOCI........ 102
DEDUCTIONS........ 105

## BOOK II.

### *RECTANGLES AND SQUARES—*

DEFINITIONS........ 112
PROPOSITIONS 1–14 WITH RIDERS........ 116

APPENDIX II.

PROPOSITIONS 1–3........ 147
LOCI........ 149
DEDUCTIONS........ 151

## BOOK III.

### *CIRCLES—*

DEFINITIONS........ 154
PROPOSITIONS 1–37 WITH RIDERS........ 162

APPENDIX III.

RADICAL AXES........ 214
DEDUCTIONS........ 218

## BOOK IV.

*CIRCLES AND THEIR SCRIBED REGULAR POLYGONS, REGULAR POLYGONS AND THEIR SCRIBED CIRCLES—*

PAGE
DEFINITIONS....................................................223
PROPOSITIONS 1–16 WITH RIDERS....................................224

### APPENDIX IV.

PROPOSITIONS 1–2...............................................250
DEDUCTIONS.....................................................256

## BOOK V.

*PROPORTION IN GENERAL—*

DEFINITIONS....................................................261
AXIOMS.........................................................265
PROPOSITIONS 1–24..............................................265
PROPOSITIONS A, B, C...........................................269
PROPOSITION D..................................................285

## BOOK VI.

*PROPORTIONAL SECTION AND SIMILAR FIGURES—*

DEFINITIONS....................................................286
PROPOSITIONS 1–33 WITH RIDERS....................................287
PROPOSITION A..................................................294
PROPOSITIONS B, C, D............................................338

### APPENDIX VI.

TRANSVERSALS...................................................343
HARMONICAL PROGRESSION.........................................347
DEDUCTIONS.....................................................352

# EUCLID'S ELEMENTS.

## BOOK I.

### DEFINITIONS.

1. **A point** has position, but it has no magnitude.

A point is indicated by a dot with a letter attached, as the point P. · P

The dots employed to represent points are not strictly geometrical points, for they have some size, else they could not be seen. But in geometry the only thing connected with a point, or its representative a dot, which we consider, is its position.

2. **A line** has position, and it has length, but neither breadth nor thickness.

Hence the ends of a line are points, and the intersection of two lines is a point.

A line is indicated by a stroke with a letter attached, as the line *C*. C———

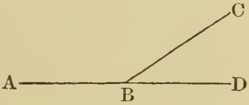
Oftener, however, a letter is placed at each end of the line, as the line *AB*. A————B

The strokes, whether of pen or pencil, employed to represent lines, are not strictly geometrical lines, for they have some breadth and some thickness. But in geometry the only things connected with a line which we consider, are its position and its length.

3. If two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a **straight line**.

Hence two straight lines cannot inclose a space, nor can they have any part in common.

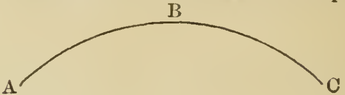
Thus the two lines *ABC* and *ABD*, which have the part *AB* in common, cannot both be straight lines.



Euclid's definition of a straight line is 'that which lies evenly to the points within itself.'

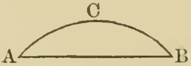
4. **A curved line**, or a **curve**, is a line of which no part is straight.

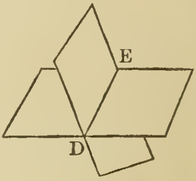
Thus *ABC* is a curve.



5. **A surface** (or superficies) has position, and it has length and breadth, but not thickness.

Hence the boundaries of a surface, and the intersection of two surfaces, are lines. Thus *AB*, *ACB*, and *DE* are lines.



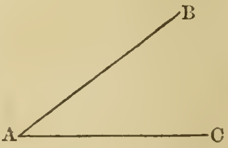


6. **A plane surface** (or a plane) is such that if any two points whatever be taken on it, the straight line joining them lies wholly in that surface.

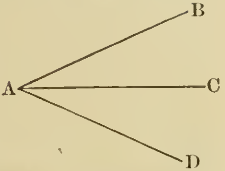
This definition (which is not Euclid's, but is due to Heron of Alexandria) affords the practical test by which we ascertain whether a given surface is a plane or not. We take a piece of wood or iron with one of its edges straight, and apply this edge in various positions to the surface. If the straight edge fits closely to the surface in every position, we conclude that the surface is plane.

7. When two straight lines are drawn from the same point, they are said to contain a **plane angle**. The straight lines are called the **arms** of the angle, and the point is called the **vertex**.

Thus the straight lines *AB*, *AC* drawn from *A* are said to contain the angle *BAC* ; *AB* and *AC* are the arms of the angle, and *A* is the vertex.



An angle is sometimes denoted by three letters, but these letters must be placed so that the one at the vertex shall always be between the other two. Thus the given angle is called *BAC* or *CAB*, never *ABC*, *ACB*, *CBA*, *BCA*. When only one angle is formed at a vertex it is often denoted by a single letter, that letter, namely, at the vertex. Thus the given angle may be called the angle $A$. But when there are several angles at the same vertex, it is necessary, in order to avoid ambiguity, to use three letters to express the angle intended. Thus, in the annexed figure, there are three angles at the vertex $A$, namely, $BAC$, $CAD$, $BAD$.



Sometimes the arms of an angle have several letters attached to them; in which case the angle may be denoted in various ways.

Fig. 1.

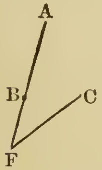


Fig. 2.

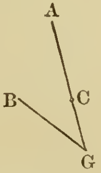
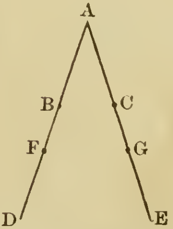
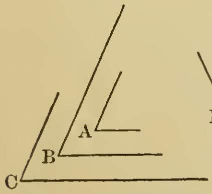


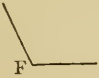
Fig. 3.

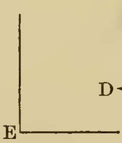


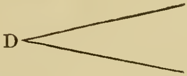
Thus the angle $F$ (fig. 1) may be called $AFC$ or $BFC$ indifferently; the angle $G$ (fig. 2) may be called $AGB$ or $CGB$; the angle $A$ (fig. 3) may be called $BAC$, $FAG$, $DAE$, $FAC$, $GAB$, and so on.

It is important to observe that all these ways of denoting any particular angle do not alter the angle; for example, the angle $BAC$ (fig. 3) is not made any larger by calling it the angle $FAG$, or the angle $DAE$. In other words, the size of an angle does not depend on the length of its arms; and hence, if the two arms of one angle are respectively equal to the two arms of another angle, the angles themselves are not necessarily equal.



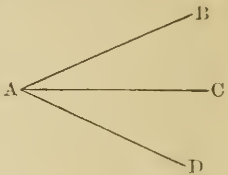






As a further illustration, the angles $A$, $B$, $C$ with unequal arms are all equal; of the angles $D$, $E$, $F$, that with the shortest arms is the largest, and that with the longest arms is the smallest.

8. If three straight lines are drawn from the same point, three different angles are formed. Thus $AB$, $AC$, $AD$, drawn from $A$, form the three angles $BAC$, $CAD$, $BAD$.



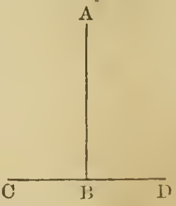
The angles $BAC$, $CAD$, which have a common arm $AC$, and lie on opposite sides of it, are called **adjacent** angles; and the angle $BAD$, which is equal to angle $BAC$ and angle $CAD$ added together, is called the **sum** of the angles $BAC$ and $CAD$. Since the angle $BAD$ is obtained by adding together the two angles $BAC$ and $CAD$, the angle $CAD$ will be obtained by subtracting the angle $BAC$ from the angle $BAD$; and similarly the angle $BAC$ will be obtained by subtracting the angle $CAD$ from the angle $BAD$. Hence the angle $CAD$ is called the **difference** of the angles $BAD$ and $BAC$; and the angle $BAC$ is called the **difference** of the angles $BAD$ and $CAD$.

9. The **bisector** of an angle is the straight line that divides it into two equal angles.

Thus (see preceding fig.), if angle $BAC$ is equal to angle $CAD$, $AC$ is called the bisector of angle $BAD$.

The word *bisect*, in Mathematics, means always, to cut into two *equal* parts.

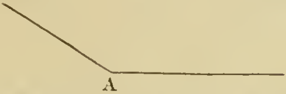
10. When a straight line stands on another straight line, and makes the adjacent angles equal to each other, each of the angles is called a **right** angle; and the straight line which stands on the other is called a **perpendicular** to it.



Thus, if $AB$ stands on $CD$ in such a manner that the adjacent angles $ABC$, $ABD$ are equal to one another, then these angles are called right angles, and *AB* is said to be perpendicular to *CD*.

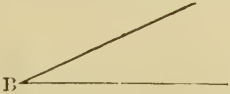
11. An **obtuse** angle is one which is greater than a right angle.

Thus *A* is an obtuse angle.



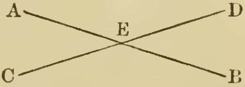
12. An **acute** angle is one which is less than a right angle.

Thus *B* is an acute angle.



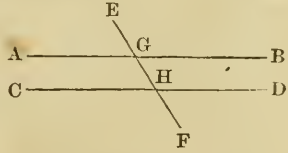
13. When two straight lines intersect each other, the opposite angles are called **vertically opposite angles.**

Thus *AEC* and *BED* are vertically opposite angles; and so are *AED* and *BEC*.



14. **Parallel** straight lines are such as are in the same plane, and being produced ever so far both ways do not meet.

Thus *AB* and *CD* are parallel straight lines.

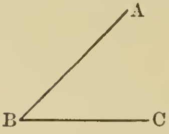


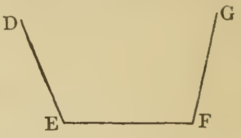
If a straight line *EF* intersect two parallel straight lines *AB*, *CD*, the angles *AGH*, *GHD* are called **alternate** angles, and so are angles *BGH*, *GHC*; angles *AGE*, *BGE*, *CHF*, *DHF* are called **exterior** angles, and the **interior opposite** angles corresponding to these are *CHG*, *DHG*, *AGH*, *BGH*.

15. **A figure** is that which is inclosed by one or more boundaries; and a **plane figure** is one bounded by a line or lines drawn upon a plane.

The space contained within the boundary of a plane figure is called its **surface**; and its surface in reference to that of another figure, with which it is compared, is called its **area**.

The word *figure*, as here defined, is restricted to closed figures Thus *ABC*, *DEFG*, according to the definition, would not be figures. The word is, however, very frequently used in a wider sense to mean any combination of points, lines, or surfaces.



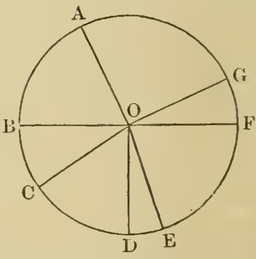


16. **A circle** is a plane figure contained by one (*curved*) line which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the **centre** of the circle.

Thus *ABCDEFG* is a circle, if all the straight lines which can be drawn from *O* to the circumference, such as *OA*, *OB*, *OC*, &c., are equal to one another; and *O* is the centre of the circle.

Strictly speaking, a circle is an inclosed space or surface, and the circumference is the line which incloses it. Frequently, however, the word circle is employed instead of circumference.

It is usual to denote a circle by three letters placed at points on its circumference. The reason for this will appear later on.



17. **A radius** (plural, *radii*) of a circle is a straight line drawn from the centre to the circumference.

Thus *OA*, *OB*, *OC*, &c. are radii of the circle *ACF*.

18. **A diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

Thus in the preceding figure *BF* is a diameter of the circle *ACF*.

## RECTILINEAL FIGURES.

19. **Rectilineal** figures are those which are contained by straight lines.

The straight lines are called **sides**, and the sum of all the sides is called the **perimeter** of the figure.

20. Rectilineal figures contained by three sides are called **triangles**.

21. Rectilineal figures contained by four sides are called **quadrilaterals**.

22. Rectilineal figures contained by more than four sides are called **polygons**.

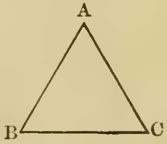
Sometimes the word polygon is used to denote a rectilineal figure of any number of sides, the triangle and the quadrilateral being included.

## CLASSIFICATION OF TRIANGLES.

*First, according to their sides*—

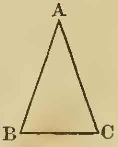
23. An **equilateral** triangle is one that has three equal sides.

Thus, if $AB$, $BC$, $CA$ are all equal, the triangle $ABC$ is equilateral.



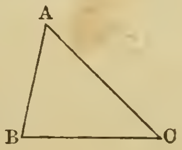
24. An **isosceles** triangle is one that has two equal sides.

Thus, if $AB$ is equal to $AC$, the triangle $ABC$ is isosceles.



25. A **scalene** triangle is one that has three unequal sides.

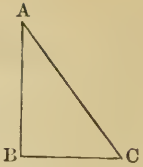
Thus, if $AB$, $BC$, $CA$ are all unequal, the triangle $ABC$ is scalene.



*Second, according to their angles—*

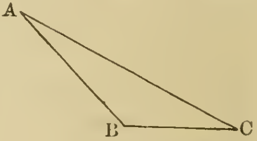
26. A **right-angled** triangle is one that has a right angle.

Thus, if $ABC$ is a right angle, the triangle $ABC$ is right-angled.



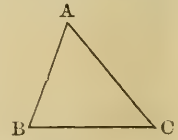
27. An **obtuse-angled** triangle is one that has an obtuse angle.

Thus, if $ABC$ is an obtuse angle, the triangle $ABC$ is obtuse-angled.



28. An **acute-angled** triangle is one that has three acute angles.

Thus, if angles $A$, $B$, $C$ are each of them acute, the triangle $ABC$ is acute-angled.



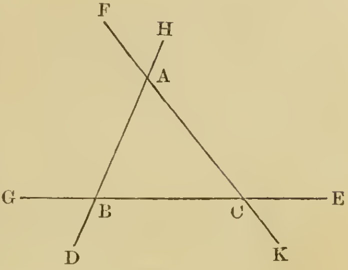
29. Any side of a triangle may be called the **base.** In an isosceles triangle, the side which is neither of the equal sides is usually called the **base.** In a right-angled triangle, one of the sides which contain the right angle is often called the **base,** and the other the **perpendicular**; the side opposite the right angle is called the **hypotenuse.**

Any of the angular points of a triangle may be called a **vertex.** If one of the sides of a triangle has been called the base, the angular point opposite that side is usually called the vertex.

Thus, if $BC$ is called the base of a triangle $ABC$, $A$ is the vertex.

30. If the sides of a triangle be prolonged both ways, nine angles are formed in addition to the angles of the triangle.

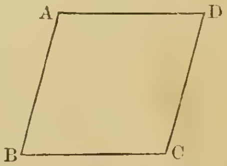
Thus at the point *A* there are the angles *CAH*, *HAF*, *FAB;* at *B*, the angles *ABG*, *GBD*, *DBC;* at *C*, the angles *BCK*, *KCE*, *ECA*.



Of these nine, six only are called **exterior** angles, the three which are not so called being *HAF*, *GBD*, *KCE*. Angles *ABC*, *BCA*, *CAB* are sometimes called the **interior** angles of the triangle.

### CLASSIFICATION OF QUADRILATERALS.

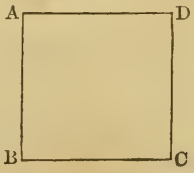
31. **A rhombus** is a quadrilateral that has all its sides equal.



Thus, if *AB*, *BC*, *CD*, *DA* are all equal, the quadrilateral *ABCD* is a rhombus. The rhombus *ABCD* is sometimes named by two letters placed at opposite corners, as *AC* or *BD*.

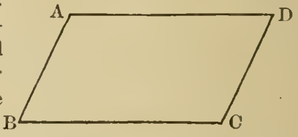
Euclid defines a rhombus to be 'a quadrilateral that has all its sides equal, but its angles not right angles.'

32. **A square** is a quadrilateral that has all its sides equal, and all its angles right angles.



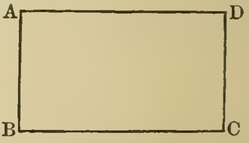
Thus, if *AB*, *BC*, *CD*, *DA* are all equal, and the angles *A*, *B*, *C*, *D* right angles, the quadrilateral *ABCD* is a square. The square *ABCD* is sometimes named by two letters placed at opposite corners, as *AC* or *BD;* and it is said to be described on any one of its four sides.

33. **A parallelogram** is a quadrilateral whose opposite sides are parallel.



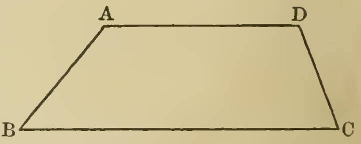
Thus, if *AB* is parallel to *CD*, and *AD* parallel to *BC*, the quadrilateral *ABCD* is a parallelogram. The parallelogram *ABCD* is sometimes named by two letters placed at opposite corners, as *AC* or *BD*; and any one of its four sides may be called the base on which it stands.

34. **A rectangle** is a quadrilateral whose opposite sides are parallel, and whose angles are right angles.



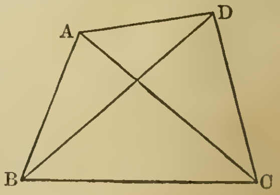
Thus, if *AB* is parallel to *CD*, *AD* parallel to *BC*, and the angles *A*, *B*, *C*, *D* right angles, the quadrilateral *ABCD* is a rectangle. The rectangle *ABCD* is sometimes named by two letters placed at opposite corners, as *AC* or *BD*. In books on mensuration, *BC* and *AB* would be called the length and the breadth of the rectangle. The definitions of a square and a rectangle are somewhat redundant—that is, more is said about a square and a rectangle than is absolutely necessary to distinguish them from other quadrilaterals. This will be seen later on.

35. **A trapezium** is a quadrilateral that has two sides parallel.



Thus, if *AD* is parallel to *BC*, the quadrilateral *ABCD* is a trapezium. The word *trapezoid* is sometimes used instead of trapezium.

36. **A diagonal** of a quadrilateral is a straight line joining any two opposite corners.



Thus *AC* and *BD* are diagonals of the quadrilateral *ABCD*.

## POSTULATES.

Let it be granted :

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length either way.

3. That a circle may be described with any centre, and at any distance from that centre.

The three postulates may be considered as stating the only instruments we are allowed to use in elementary geometry. These are the *ruler* or *straight-edge*, for drawing straight lines, and the *compasses*, for describing circles. The ruler is not to be divided at its edge (or graduated), so as to enable us to measure off particular lengths; and the compasses are to be employed in describing circles only when the centre of the circle is at one given point, and the circumference must pass through another given point. Neither ruler nor compasses can be used to carry distances.

If two points $A$ and $B$ are given, and we wish to draw a straight line from $A$ to $B$, it is usual to say simply 'join $AB$.' To *produce* a straight line, means not to make a straight line when there is none, but when there is a straight line already, to make it longer. The third postulate is sometimes expressed, 'a circle may be described with any centre and any radius.' That, however, is not to be taken as meaning with a radius equal to any given straight line, but only with a radius equal to any given straight line drawn from the centre.

[The restrictions imposed on the use of the ruler and the compasses, somewhat inconsistently on Euclid's part, are never adhered to in practice.]

## AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the sums are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the sums are unequal, the greater sum being obtained from the greater unequal.

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Things which are doubles of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. The whole is greater than its part, and equal to the sum of all its parts.

9. Magnitudes which coincide with one another are equal to one another.

10. All right angles are equal to one another.

11. Two straight lines which intersect one another cannot be both parallel to the same straight line.

An axiom is a self-evident truth, or it is a statement the truth of which is admitted at once and without demonstration. Some of Euclid's axioms are general—that is, they apply to magnitudes of all kinds, and not to geometrical magnitudes only. The first axiom, which says that things which are equal to the same thing are equal to one another, applies not only to lines, angles, surfaces, and solids, but also, for example, to numbers, which are arithmetical, and to forces, which are physical, magnitudes. It will be seen that the first eight axioms are general, and that the last three are geometrical.

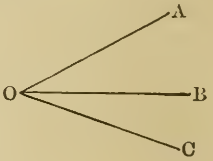
It ought, perhaps, to be noted that some of the axioms are often applied, not in the general form in which they are stated, but in particular cases that come under the general form. For example, under the general form of Axiom 2 would come two particular cases: If equals be added to the same thing, the sums are equal; and If the same thing be added to equals, the sums are equal. Again, a particular case coming under the general form of Axiom 4 would be: If the same thing be added to unequals, the sums are unequal, the greater sum being obtained from the greater unequal. Axioms 6 and 7, on the other hand, are only particular cases of more general ones—namely, Things which are double of equals are equal, and Things which are halves of equals are equal; and these axioms again are only particular cases of still more general ones: Similar multiples of equals (or of the same thing) are equal, and Similar fractions of equals (or of the same thing) are equal.

Axiom 9 is often called Euclid's definition or test of equality; and the method of ascertaining whether two magnitudes are equal by seeing whether they coincide—that is, by mentally applying the one to the other, is called the method of superposition. Two magnitudes (for example, two triangles) which coincide are said to be *congruent;* and this word, if it is thought desirable, may be used instead of the phrase, 'equal in every respect.' Axiom 10 is, strictly speaking, a proposition capable of proof. The proof is not given here, as at this stage it would perhaps not be fully appreciated by the pupil. After he has read and understood the definitions of the third book, he will probably be able to prove it for himself. Axiom 11, frequently referred to as Playfair's axiom (though Playfair states that it is assumed by others, particularly by Ludlam in his *Rudiments of Mathematics*), has been substituted for that given by Euclid, which is proved as a corollary to Proposition 29.

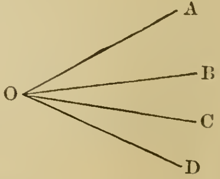
## QUESTIONS ON THE DEFINITIONS, POSTULATES, AXIOMS.

1. How do we indicate a point?
2. What is the only thing that a point has? What has it not?
3. Could a number of geometrical points placed close to one another form a line? Why?
4. Draw two lines intersecting each other in two points.
5. Could two straight lines be drawn intersecting each other in two points?
6. What is Euclid's definition of a 'straight' line?
7. Could a number of geometrical lines placed close to one another form a surface? Why?
8. When two points are taken on a plane surface, and a straight line is drawn from the one to the other, where will the straight line lie?
9. If a straight line is drawn on a plane surface and then produced, where will the produced part lie?

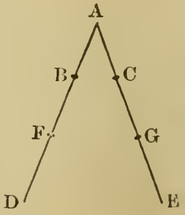
10. Would it be possible to draw a straight line upon a surface that was not plane? If so, give an example.
11. How many arms has an angle?
12. What name is given to the point where the arms meet?
13. When an angle is denoted by three letters, may the letters be arranged in any order?
14. If not, in how many ways may they be arranged, and what precaution must be observed?



15. When is it necessary to name an angle by three letters?
16. How else may an angle be named?
17. *OA*, *OB*, *OC* are three straight lines which meet at *O*. Name the three angles which they form.
18. Name the angle contained by *OA* and *OB*; by *OB* and *OC*; by *OC* and *OA*.



19. *OA*, *OB*, *OC*, *OD* are four straight lines which meet at *O*. Name the six angles which they form.
20. Name the angle contained by *OA* and *OB*; by *OB* and *OC*; by *OC* and *OD*; by *OA* and *OC*; by *OB* and *OD*; by *OA* and *OD*.

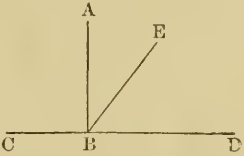


21. Write down all the ways in which the angle *A* can be named.
22. If the arms of one angle are respectively equal to the arms of another angle, what inference can we draw regarding the sizes of the angles?
23. In the figure to Question 17, if the angles *AOB* and *BOC* are added together, what angle do they form?
24. In the same figure, if the angle *AOB* is taken away from the angle *AOC*, what angle is left?
25. In the same figure, if the angle *BOC* is taken away from the angle *AOC*, what angle is left?
26. The following questions refer to the figure to Question 19:
    - (*a*) Add together the angles *AOB* and *BOC*; *AOB* and *BOD*; *AOC* and *COD*; *BOC* and *COD*.
    - (*b*) From the angle *AOD* subtract successively the angles *COD*, *AOB*, *AOC*, *BOD*.

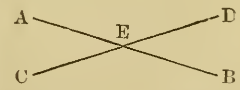
(*c*) From the angle $BOD$ subtract the angles $COD$, $BOC$.

(*d*) To the sum of the angles $AOB$ and $BOC$ add the difference of the angles $BOD$ and $BOC$; and from the sum of $AOB$ and $BOC$ subtract the difference of $BOD$ and $COD$.

27. Draw, as well as you can, two equal angles with unequal arms.
28. " " two unequal " equal "
29. If two adjacent angles are equal, must they necessarily be right angles? Draw a figure to illustrate your answer.
30. If two adjacent angles are equal, what name could be given to the arm that is common to the two angles?
31. When an angle is greater than a right angle, what is it called?
32. " less " "
33. " equal to " "
34. In the accompanying figure, name two right angles, two acute angles, and one obtuse angle.

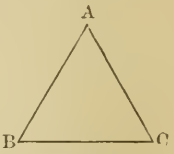


35. What are angles $AEC$, $AED$ called with reference to each other? angles $AEC$, $BED$? angles $AEC$, $BEC$? angles $BEC$, $AED$? angles $BEC$, $BED$?

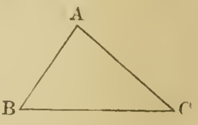


36. Would it be a sufficient definition of parallel straight lines to say that they never meet though produced indefinitely far either way? Illustrate your answer by reference to the edges of a book, or otherwise.
37. Draw three straight lines, every two of which are parallel.
38. Draw three straight lines, only two of which are parallel.
39. Draw three straight lines, no two of which are parallel.
40. What is the least number of lines that will inclose a space? Illustrate your answer by an example.
41. How many radii of a circle are equal to one diameter?
42. How do we know that all radii of a circle are equal?
43. Prove that all diameters of a circle are equal.
44. Are all lines drawn from the centre of a circle to the circumference equal to one another?
45. What is the distinction between a circle and a circumference?
46. Is the one word ever used for the other?

47. How many letters are generally used to denote a circle?
48. Would it be a sufficient definition of a diameter of a circle to say that it consists of two radii?
49. Prove that the distance of a point inside a circle from the centre is less than a radius of the circle.
50. Prove that the distance of a point outside a circle from the centre is greater than a radius of the circle.
51. What is the least number of straight lines that will inclose a space?
52. What name is given to figures that are contained by straight lines?
53. Could three straight lines be drawn so that, even if they were produced, they would not inclose a space?
54. What is the least number of sides that a rectilineal figure can have?

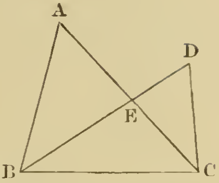


55. $ABC$ is a triangle. Name it in five other ways.
56. If $AB$ is equal to $AC$, what is triangle $ABC$ called?
57. If $AB$, $BC$, $CA$ are all equal, what is triangle $ABC$ called?
58. If $AB$, $BC$, $CA$ are all unequal, what is triangle $ABC$ called?
59. What name is given to the sum of $AB$, $BC$, and $CA$?
60. Which side of a triangle is called the base?
61. Which side of an isosceles triangle is called the base?
62. When the hypotenuse of a triangle is mentioned, of what sort must the triangle be?
63. What names are sometimes given to those sides of a right-angled triangle which contain the right angle?
64. Would it be a sufficient definition of an acute-angled triangle to say that it had neither a right nor an obtuse angle?



65. $ABC$ is a triangle. Name by one letter the angles respectively opposite to the sides $AB$, $BC$, $CA$.
66. Name by three letters the angles respectively opposite to the sides $AB$, $BC$, $CA$.
67. Name the sides respectively opposite to the angles $A$, $B$, $C$.
68. Name by one letter and by three letters the angle contained by $AB$ and $AC$; by $AB$ and $BC$; by $AC$ and $BC$.

69. Name all the triangles in the accompanying figure.



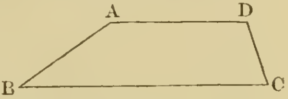
70. Name the additional triangles that would be formed if *AD* were joined.

71. Name by three letters all the angles opposite to *BC;* to *BE;* to *CE.*

72. Name all the sides that are opposite to angle *A;* to angle *D.*

73. Name all the angles in the figure that are called exterior angles of the triangle *BEC;* of the triangle *AEB:* of the triangle *CED.*

74. *ABCD* is a quadrilateral. Name it in seven other ways.



75. If the diagonals *AC*, *BD* be drawn, and *E* be their point of intersection, how many triangles will there be in the diagram? Name them.

76. Name the two angles opposite to the diagonal *AC.*

77. " " " *BD.*

78. " through which the diagonal *AC* passes.

79. " " " " *BD* "

80. Could a square, with propriety, be called a rhombus?

81. Could a rhombus be called a square?

82. Could a rectangle be called a parallelogram?

83. Could a parallelogram be called a rectangle?

84. Would it be a sufficient definition of a parallelogram to say that it is a figure whose opposite sides are parallel? Why?

85. Could a parallelogram or a rectangle be called a trapezium?

86. Could a trapezium be called a parallelogram or a rectangle?

87. What is a diagonal of a quadrilateral, and how many diagonals has a quadrilateral?

88. How many sides has a polygon?

89. Which postulate allows us to join two points?

90. " " produce a straight line?

91. " " describe a circle?

92. In what sense is the word 'circle' used in the third postulate?

93. What are the only instruments that may be used in elementary plane geometry? Under what restrictions are they to be used?

94. What is an axiom? Give an example of one.

95. State Euclid's axiom about magnitudes which coincide.

96. Would it be correct to say, magnitudes which fill the same space, instead of magnitudes which coincide? Illustrate your answer by reference to straight lines, and angles.
97. What is Euclid's axiom about right angles?
98. What is the axiom about parallels?
99. Would it be correct to say, two straight lines which pass through the same point cannot be both parallel to the same straight line?
100. Could two straight lines which do not pass through the same point be both parallel to a third straight line?

---

## EXPLANATION OF TERMS.

Propositions are divided into two classes, theorems and problems.

A **theorem** is a truth that requires to be proved by means of other truths already known. The truths already known are either axioms or theorems.

A **problem** is a construction which is to be made by means of certain instruments. The instruments allowed to be used are (see the remarks on the postulates) the ruler and the compasses.

A **corollary** is a truth which is (more or less) easily inferred from a proposition.

In the statement of a theorem there are two parts, the **hypothesis** and the **conclusion**. Thus, in the theorem, 'If two sides of a triangle be equal, the angles opposite to them shall be equal,' the part, 'if two sides of a triangle be equal,' is the hypothesis, or that which is assumed; the other part, 'the angles opposite to them shall be equal,' is the conclusion, or that which is inferred from the hypothesis.

The **converse** of a theorem is derived from the theorem by interchanging the hypothesis and the conclusion. Thus, the converse of the theorem mentioned above is, 'If in a triangle the angles opposite two sides be equal, the sides shall be equal.'

When the hypothesis of a theorem consists of several hypotheses, there may be more than one converse to the theorem.

In proving propositions, recourse is sometimes had to the following method. The proposition is supposed not to be true, and the consequences of this supposition are then examined, till at length a result is reached which is impossible or absurd. It is therefore inferred that the proposition must be true. Such a method of proof is called an **indirect** demonstration, or sometimes a **reductio ad absurdum** (a reducing to the absurd).

## SYMBOLS AND ABBREVIATIONS.

$+$, read *plus*, is the sign of addition, and signifies that the magnitudes between which it is placed are to be added together.

$-$, read *minus*, is the sign of subtraction, and signifies that the magnitude written after it is to be subtracted from the magnitude written before it.

$\sim$, read *difference*, is sometimes used instead of minus, when it is not known which of the two magnitudes before and after it is the greater.

$=$ is the sign of equality, and signifies that the magnitudes between which it is placed are equal to each other. It is used here as an abbreviation for 'is equal to,' 'are equal to,' 'be equal to, and 'equal to.'

$\perp$ stands for 'perpendicular to,' or 'is perpendicular to.'

$\parallel$ " 'parallel to,' or 'is parallel to.'

$\angle$ " 'angle.'

$\triangle$ " 'triangle.'

$\|^{m}$ " 'parallelogram.'

$\odot$ " 'circle.'

$\bigcirc^{ce}$ " 'circumference.'

$\therefore$ " 'therefore.' This symbol turned upside down ($\because$), which is sometimes used for 'because' or 'since,' I have not introduced, partly because some writers use it for 'therefore,' and partly because it is easily confounded with the other.

$AB^2$ stands for 'the square described on $AB$.'

$AB \cdot BC$ stands for 'the rectangle contained by $AB$ and $BC$.'

$A : B$ stands for 'the ratio of $A$ to $B$.'

$\left\{\begin{matrix} A : B \\ B : C \end{matrix}\right\}$ stands for 'the ratio compounded of the ratios of $A$ to $B$ and $B$ to $C$.'

$A : B = C : D$ stands for the proportion ‘$A$ is to $B$ as $C$ is to $D$.’
The small letters $a$, $b$, $c$, $m$, $n$, $p$, &c. stand for numbers.

| | | |
|---|---|---|
| *App.* | stands for | ‘appendix.’ |
| *Ax.* | " | ‘axiom.’ |
| *Const.* | " | ‘construction.’ |
| *Cor.* | " | ‘corollary.’ |
| *Def.* | " | ‘definition.’ |
| *Hyp.* | " | ‘hypothesis.’ |
| *Post.* | " | ‘postulate.’ |
| *Rt.* | " | ‘right.’ |

In the references given at the right-hand side of the page (Euclid gives no references), the Roman numerals indicate the number of the book, the Arabic numerals the number of the proposition. Thus, I. 47 means the forty-seventh proposition of the first book.

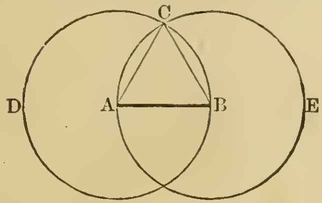
In the figures to certain of the theorems, it will be seen that some lines are *thick*, and some *dotted*. The thick lines are those which are given, the dotted lines are those which are drawn in order to prove the theorem. [In a few figures this arrangement has been neglected to attain another object.]

In the figures to certain of the problems, some lines are *thick*, some *thin*, and some *dotted*. The thick lines are those which are given, the thin lines are those which are drawn in order to effect the construction, and the dotted lines are those which are necessary for the proof that the construction is correct.

In the figures which illustrate definitions, the lines are almost invariably thin.

## PROPOSITION 1. PROBLEM.

*To describe an equilateral triangle on a given straight line.*



Let $AB$ be the given straight line :
*it is required to describe an equilateral triangle on* $AB$.

With centre $A$ and radius $AB$, describe ⊙ $BCD$. *Post.* 3
With centre $B$ and radius $BA$, describe ⊙ $ACE$; *Post.* 3
and let the two circles intersect at $C$.
Join $AC$, $BC$. *Post.* 1

$ABC$ shall be an equilateral triangle.

For $AB = AC$, being radii of the ⊙ $BCD$; *I. Def.* 16
and $AB = BC$, being radii of the ⊙ $ACE$; *I. Def.* 16
∴ $AC = BC$. *I. Ax.* 1
∴ $AB$, $AC$, $BC$ are all equal,
and $ABC$ is an equilateral triangle. *I. Def.* 23

### DEDUCTIONS.

1. If the two circles intersect also at $F$, and $AF$, $BF$ be joined, prove that $ABF$ is an equilateral triangle.
2. Show how to find a point which is equidistant from two given points.
3. Show how to make a rhombus having one of its diagonals equal to a given straight line.
4. Show how to make a rhombus having each of its sides equal to a given straight line.

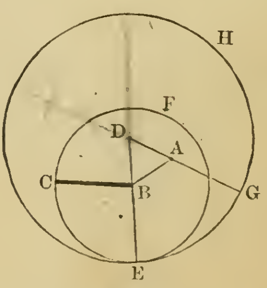
5. If $AB$ be produced both ways to meet the two circles again at $D$ and $E$, prove that the straight line $DE$ is equal to the sum of the three sides of the triangle $ABC$.
6. Show how to find a straight line equal to the sum of the three sides of any triangle.

Show how to find a straight line which shall be :

7. Twice as great as a given straight line.
8. Thrice " " "
9. Four times " " "
10. Five " " " " &c.

## PROPOSITION 2. PROBLEM.

*From a given point to draw a straight line equal to a given straight line.*



Let $A$ be the given point, and $BC$ the given straight line: *it is required to draw from A a straight line = BC.*

Join $AB$, *Post.* 1
and on it describe the equilateral △ $DBA$. *I.* 1
With centre $B$ and radius $BC$, describe the ⊙ $CEF$; *Post.* 3
and produce $DB$ to meet the $\bigcirc^{ce}$ $CEF$ in $E$. *Post.* 2

With centre $D$, and radius $DE$, describe the ⊙ $EGH$; *Post.* 3
and produce $DA$ to meet the $\bigcirc^{ce}$ $EGH$ in $G$. *Post.* 2
$AG$ shall $= BC$.

Because $DE = DG$, being radii of ⊙ $EGH$, *I. Def.* 16
and $DB = DA$, being sides of an equilateral triangle; *I. Def.* 23
∴ remainder $BE$ = remainder $AG$. *I. Ax.* 3
But $BE = BC$, being radii of ⊙ $CEF$; *I. Def.* 16
∴ $AG = BC$. *I. Ax.* 1

1. If the radius of the large circle be double the radius of the small circle, where will the given point be?
2. $AB$ is a given straight line; show how to draw from $A$ any number of straight lines equal to $AB$.
3. $AB$ is a given straight line; show how to draw from $B$ any number of straight lines equal to $AB$.
4. $AB$ is a given straight line; show how to draw through $A$ any number of straight lines double of $AB$.
5. $AB$ is a given straight line; show how to draw through $B$ any number of straight lines double of $AB$.
6. On a given straight line as base, describe an isosceles triangle each of whose sides shall be equal to a given straight line.

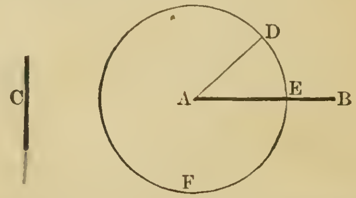
   May the second given straight line be of any size? If not, how large or how small may it be?

Give the construction and proof of the proposition—

7. When the equilateral triangle $ABD$ is described on that side of $AB$ opposite to the one given in the text.
8. When the equilateral triangle $ABD$ is described on the same side of $AB$ as in the text, but when its sides are produced through the vertex and not beyond the base.
9. When the equilateral triangle $ABD$ is described on that side of $AB$ opposite to the one given in the text, and when its sides are produced through the vertex.
10. When the given point $A$ is joined to $C$ instead of $B$. Make diagrams for all the cases that can arise by describing the equilateral triangle on either side of $AC$, and producing its sides either beyond the base or through the vertex.

## PROPOSITION 3. PROBLEM.

*From the greater of two given straight lines to cut off a part equal to the less.*



Let $AB$ and $C$ be the two given straight lines, of which $AB$ is the greater:

*it is required to cut off from $AB$ a part $= C$.*

From $A$ draw the straight line $AD = C$: *I. 2*

with centre $A$ and radius $AD$, describe the $\odot$ $DEF$, *Post. 3*

cutting $AB$ at $E$. $AE$ shall $= C$.

For $AE = AD$, being radii of $\odot$ $DEF$. *I. Def. 15*

But $AD = C$; *Const.*

$\therefore$ $AE = C$. *I. Ax. 1*

1. Give the construction and the proof of this proposition, using the point $B$ instead of the point $A$.
2. Produce the less of two given straight lines so that it may be equal to the greater.
3. If from $AB$ (fig. 1 and fig. 2) there be cut off $AD$ and $BE$, each equal to $C$, prove $AE = BD$.

Fig. 1.

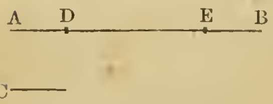
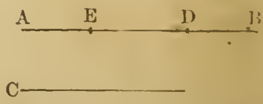


Fig. 2.



4. Show how to find a straight line equal to the sum of two given straight lines.

5. Show how to find a straight line equal to the difference of two given straight lines.
6. Show that if the difference of two straight lines be added to the sum of the two straight lines, the result will be double of the greater straight line.
7. Show that if the difference of two straight lines be taken away from the sum of the two straight lines, the result will be double of the less straight line.

---

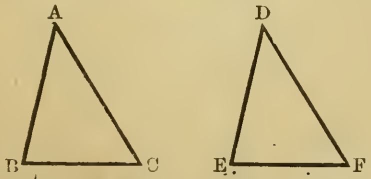
## PROPOSITION 4. THEOREM.

*If two sides and the contained angle of one triangle be equal to two sides and the contained angle of another triangle, the two triangles shall be equal in every respect—that is,*

(1) *The third sides shall be equal,*

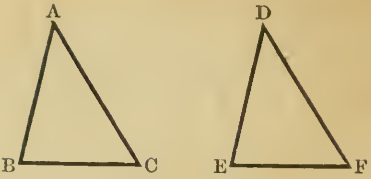
(2) *The remaining angles of the one triangle shall be equal to the remaining angles of the other triangle,*

(3) *The areas of the two triangles shall be equal.*



In $\triangle$s $ABC, DEF$, let $AB = DE$, $AC = DF$, $\angle A = \angle D$: *it is required to prove* $BC = EF$, $\angle B = \angle E$, $\angle C = \angle F$, $\triangle ABC = \triangle DEF$.

If $\triangle ABC$ be applied to $\triangle DEF$,
so that $A$ falls on $D$, and so that $AB$ falls on $DE$;
then $B$ will coincide with $E$, because $AB = DE$. *Hyp.*
And because $AB$ coincides with $DE$, and $\angle A = \angle D$, *Hyp.*
$\therefore$ $AC$ will fall on $DF$.
And because $AC = DF$, *Hyp.*
$\therefore$ $C$ will coincide with $F$.



Now, since $B$ coincides with $E$, and $C$ with $F$,
$\therefore$ $BC$ will coincide with $EF$; *I. Def.* 3
$\therefore$ $BC = EF$. *I. Ax.* 9
Hence also $\angle B$ will coincide with $\angle E$;
$\therefore$ $\angle B = \angle E$; *I. Ax.* 9
and $\angle C$ will coincide with $\angle F$; $\therefore$ $\angle C = \angle F$; *I. Ax.* 9
and $\triangle ABC$ will coincide with $\triangle DEF$;
$\therefore$ $\triangle ABC = \triangle DEF$. *I. Ax.* 9

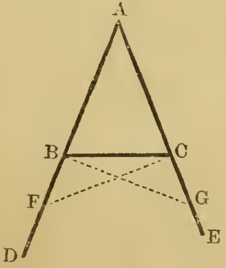
In the two $\triangle$s $ABC$, $DEF$,

1. If $AB = DE$, $AC = DF$, but $\angle A$ greater than $\angle D$, where would $AC$ fall when $ABC$ is applied to $DEF$ as in the proposition?
2. If $AB = DE$, $AC = DF$, but $\angle A$ less than $\angle D$, where would $AC$ fall?
3. If $AB = DE$, $\angle A = \angle D$, but $AC$ greater than $DF$, where would $C$ fall?
4. If $AB = DE$, $\angle A = \angle D$, but $AC$ less than $DF$, where would $C$ fall?
5. Prove the proposition beginning the superposition with the point $B$ or the point $C$ instead of the point $A$.
6. If the straight line $CD$ bisect the straight line $AB$ perpendicularly, prove any point in $CD$ equidistant from $A$ and $B$.
7. $CA$ and $CB$ are two equal straight lines drawn from the point $C$, and $CD$ is the bisector of $\angle ACB$. Prove that any point in $CD$ is equidistant from $A$ and $B$.
8. The straight line that bisects the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.
9. $ABCD$ is a quadrilateral, one of whose diagonals is $BD$. If $AB = CB$, and $BD$ bisects $\angle ABC$, prove that $AD = CD$, and that $BD$ bisects also $\angle ADC$.
10. Prove that the diagonals of a square are equal.

11. $ABCD$ is a square. $E$, $F$, $G$, $H$ are the middle points of $AB$, $BC$, $CD$, $DA$, and $EF$, $FG$, $GH$, $HE$ are joined. Prove that $EFGH$ has all its sides equal.
12. Prove by superposition that the squares described on two equal straight lines are equal.
13. If two quadrilaterals have three consecutive sides and the two contained angles in the one respectively equal to three consecutive sides and the two contained angles in the other, the quadrilaterals shall be equal in every respect.

## PROPOSITION 5. THEOREM.

*The angles at the base of an isosceles triangle are equal; and if the equal sides be produced, the angles on the other side of the base shall also be equal.*



In $\triangle\ ABC$, let $AB = AC$, and let $AB$, $AC$ be produced to $D$ and $E$:

*it is required to prove* $\angle\ ABC = \angle\ ACB$ *and* $\angle\ DBC = \angle\ ECB$.

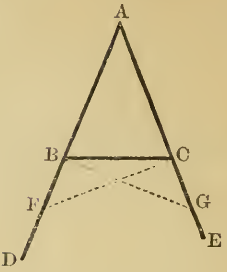
In $BD$ take any point $F$,

and from $AE$ cut off $AG = AF$; *I.* 3

join $BG$, $CF$. *Post.* 1

$$(1)\ \text{In } \triangle\text{s } AFC,\ AGB, \left\{\begin{array}{ll} FA = GA & \textit{Const.} \\ AC = AB & \textit{Hyp.} \\ \angle\ FAC = \angle\ GAB; & \end{array}\right.$$

$\therefore\ FC = GB$, $\angle\ AFC = \angle\ AGB$, $\angle\ ACF = \angle\ ABG$. *I.* 4



(2) Because the whole $AF$ = whole $AG$, *Const.*
and the part $AB$ = part $AC$; *Hyp.*
∴ the remainder $BF$ = remainder $CG$. *I. Ax.* 3

(3) In △s $BFC$, $CGB$, $\begin{cases} BF = CG & \textit{Proved in } (2) \\ FC = GB & \textit{Proved in } (1) \\ \angle BFC = \angle CGB; & \textit{Proved in } (1) \end{cases}$
∴ $\angle BCF = \angle CBG$, and $\angle FBC = \angle GCB$. *I.* 4

(4) Because whole $\angle ABG$ = whole $\angle ACF$, *Proved in* (1)
and the part $\angle CBG$ = part $\angle BCF$; *Proved in* (3)
∴ the remainder $\angle ABC$ = remainder $\angle ACB$; *I. Ax.* 3
and these are the angles at the base.
But it was proved in (3) that $\angle FBC = \angle GCB$;
and these are the angles on the other side of the base.

Cor.—If a triangle have all its sides equal, it will also have all its angles equal; or, in other words, if a triangle be equilateral, it will be equiangular.

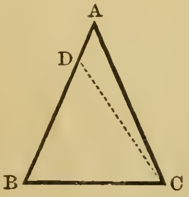
1. If two angles of a triangle be unequal, the sides opposite to them will also be unequal.
2. Two isosceles triangles $ABC$, $DBC$ stand on the same base $BC$, and on opposite sides of it; prove $\angle ABD = \angle ACD$.
3. Two isosceles triangles $ABC$, $DBC$ stand on the same base $BC$, and on the same side of it; prove $\angle ABD = \angle ACD$.
4. In the figure to the second deduction, if $AD$ be joined, prove that it will bisect the angles at $A$ and $D$.

5. $ABC$ is an isosceles triangle having $AB = AC$. In $AB$, $AC$, two points $D$, $E$ are taken equally distant from $A$; prove that the triangles $ABE$, $ACD$ are equal in all respects, and also the triangles $DBC$, $ECB$.
6. Prove that the opposite angles of a rhombus are equal.
7. $D$ and $E$ are the middle points of the sides $BC$ and $CA$ of a triangle; $DO$ and $EO$ are perpendicular to $BC$ and $CA$; show that the angles $OAB$ and $OBA$ are equal.
8. Prove the proposition by supposing the $\triangle ABC$, after leaving a trace or impression of itself, to be lifted up, turned over, and applied to the trace.
9. Prove the first part of the proposition by supposing the angle at the vertex to be bisected.

---

## PROPOSITION 6. THEOREM.

*If two angles of a triangle be equal, the sides opposite them shall also be equal.*



In $\triangle ABC$ let $\angle ABC = \angle ACB$:
*it is required to prove* $AC = AB$.

If $AC$ is not $= AB$, one of them must be the greater.
Let $AB$ be the greater;
and from it cut off $BD = AC$, *I.* 3
and join $DC$. *Post.* 1

$$\text{In } \triangle\text{s } DBC,\ ACB;\ \left\{\begin{array}{ll} DB = AC & \quad \textit{Const.} \\ BC = CB & \\ \angle DBC = \angle ACB; & \quad \textit{Hyp.} \end{array}\right.$$

$\therefore$ area of $\triangle\ DBC$ = area of $\triangle\ ACB$; *I.* 4
which is impossible, since $\triangle\ DBC$ is a part of $\triangle\ ACB$.
Hence $AC$ is not unequal to $AB$;
that is, $AC = AB$.

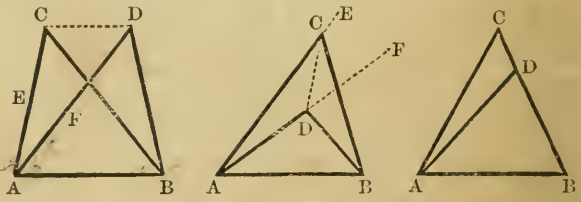
Cor.—If a triangle have all its angles equal, it will also have all its sides equal; or, in other words, if a triangle be equiangular, it will be equilateral.

1. If two sides of a triangle be unequal, the angles opposite to them will also be unequal.
2. If $ABC$ be an isosceles triangle, and if the equal angles $ABC$, $ACB$ be bisected by $BD$, $CD$, which meet at $D$; prove that $DBC$ is also an isosceles triangle.
3. In the figure to *I.* 5, if $BG$, $CF$ intersect at $H$, prove that $HBC$ is an isosceles triangle.
4. Hence prove that $FH = GH$, and that $AH$ bisects $\angle\ A$.
5. By means of what is proved in the last deduction, give a method of bisecting an angle.
6. Prove the proposition by supposing the $\triangle\ ABC$, after leaving a trace or impression of itself, to be lifted up, turned over, and applied to the trace.

---

## PROPOSITION 7. Theorem.

*Two triangles on the same base and on the same side of it cannot have their conterminous sides equal.*



If it be possible, let the two $\triangle$s $ABC$, $ABD$ on the same base $AB$, and on the same side of it, have $AC = AD$, and $BC = BD$.

Three cases may occur:

(1) The vertex of each △ may be outside the other △.

(2) The vertex of one △ may be inside the other △.

(3) The vertex of one △ may be on a side of the other △.

In the first case join *CD*; and in the second case join *CD* and produce *AC*, *AD* to *E* and *F*.

Because $AC = AD$, $\therefore \angle ECD = \angle FDC$. *I.* 5

But $\angle ECD$ is greater than $\angle BCD$; *I. Ax.* 8

$\therefore \angle FDC$ is greater than $\angle BCD$.

Much more then is $\angle BDC$ greater than $\angle BCD$.

But because $BC = BD$, $\therefore \angle BDC = \angle BCD$; *I.* 5

that is, $\angle BDC$ is greater than and equal to $\angle BCD$, which is impossible.

The third case needs no proof, because *BC* is not = *BD*.

Hence two triangles on the same base and on the same side of it cannot have their conterminous sides equal.

1. On the same base and on the same side of it there can be only one equilateral triangle.
2. On the same base and on the same side of it there can be only one isosceles triangle having its sides equal to a given straight line.
3. Two circles cannot cut each other at more than one point either above or below the straight line joining their centres.

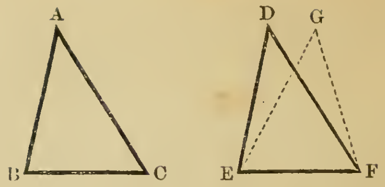
---

## PROPOSITION 8. THEOREM.

*If three sides of one triangle be respectively equal to three sides of another triangle, the two triangles shall be equal in every respect; that is,*

(1) *The three angles of the one triangle shall be respectively equal to the three angles of the other triangle,*

(2) *The areas of the two triangles shall be equal.*



In $\triangle$s $ABC$, $DEF$, let $AB = DE$, $AC = DF$, $BC = EF$: *it is required to prove* $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, *and* $\triangle ABC = \triangle DEF$.

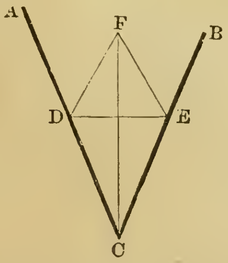
If $\triangle ABC$ be applied to $\triangle DEF$,
so that $B$ falls on $E$, and so that $BC$ falls on $EF$;
then $C$ will coincide with $F$, because $BC = EF$. *Hyp.*
Now since $BC$ coincides with $EF$,
$\therefore$ $BA$ and $AC$ must coincide with $ED$ and $DF$.
For, if they do not, but fall otherwise as $EG$ and $GF$;
then on the same base $EF$, and on the same side of it,
there will be two $\triangle$s $DEF$, $GEF$, having equal pairs
of conterminous sides,
which is impossible. *I.* 7
$\therefore$ $BA$ coincides with $ED$, and $AC$ with $DF$.
Hence $\angle A$ will coincide with $\angle D$, $\therefore$ $\angle A = \angle D$; *I. Ax.* 9
and $\angle B$ will coincide with $\angle E$, $\therefore$ $\angle B = \angle E$; *I. Ax.* 9
and $\angle C$ will coincide with $\angle F$, $\therefore$ $\angle C = \angle F$; *I. Ax.* 9
and $\triangle ABC$ will coincide with $\triangle DEF$,
$\therefore$ $\triangle ABC = \triangle DEF$. *I. Ax.* 9

1. The straight line which joins the vertex of an isosceles triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.
2. The opposite angles of a rhombus are equal.
3. Either diagonal of a rhombus bisects the angles through which it passes.
4. $ABCD$ is a quadrilateral having $AB = BC$ and $AD = DC$; prove that the diagonal $BD$ bisects the angles through which it passes, and that $\angle A = \angle C$.

5. Two isosceles triangles stand on the same base and on opposite sides of it; prove that the straight line joining their vertices bisects both vertical angles.
6. Two isosceles triangles stand on the same base and on the same side of it; prove that the straight line joining their vertices, being produced, bisects both vertical angles.
7. In the figures to the fifth and sixth deductions, prove that the straight line joining the vertices, or that straight line produced, bisects the common base perpendicularly.
8. Hence give a construction for bisecting a given straight line.
9. The diagonals of a rhombus or of a square bisect each other perpendicularly.
10. If any two circles cut each other, the straight line joining their points of intersection is bisected perpendicularly by the straight line joining their centres.
11. Prove the proposition by applying the triangles so that they may fall on opposite sides of a common base. Join the two vertices, and use I. 5 (Philon's method; see Friedlein's *Proclus*, p. 266).

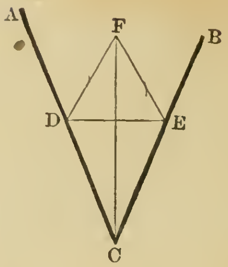
## PROPOSITION 9. PROBLEM.

*To bisect a given rectilineal angle.*



Let $ACB$ be the given rectilineal angle:
*it is required to bisect it.*

In $AC$ take any point $D$,
and from $CB$ cut off $CE = CD$. *I.* 3



Join *DE*, and on *DE*, on the side remote from *C*, describe the equilateral △ *DEF*. *I.* 1

Join *CF*. *CF* shall bisect ∠ *ACB*.

$$\text{In } \triangle\text{s } DCF,\ ECF, \left\{\begin{array}{ll} DC = EC & \qquad \textit{Const.} \\ CF = CF & \\ DF = EF; & \qquad \textit{I. Def. } 23 \end{array}\right.$$

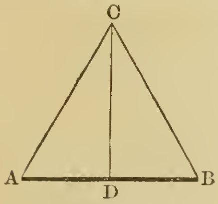
∴ ∠ *DCF* = ∠ *ECF*; *I.* 8

that is, *CF* bisects ∠ *ACB*.

1. Prove that *CF* bisects angle *DFE*.
2. If the equilateral triangle *DEF* were described on the same side of *DE* as *C* is, what three positions might *F* take?
3. Show that in one of these positions the demonstration remains the same as in the text.
4. Would an isosceles triangle *DEF* described on the base *DE* answer the purpose as well as an equilateral one? If so, why?
5. Prove the proposition and the first deduction, using I. 5 and I. 4 instead of I. 8.
6. ✓ Divide a given angle into 4 equal parts.
7. Could the number of equal parts into which an angle may be divided be extended beyond 4? If so, enumerate the numbers.
8. Prove from an equilateral triangle that if a right-angled triangle have one of the acute angles double of the other, the hypotenuse is double of the side opposite the least angle.

## PROPOSITION 10. Problem.

*To bisect a given straight line.*



Let $AB$ be the given straight line:
*it is required to bisect it.*

On $AB$ describe an equilateral △ $ABC$, I. 1
and bisect ∠ $ACB$ by $CD$, which meets $AB$ at $D$. I. 9

$AB$ shall be bisected at $D$.

$$\text{In } \triangle\text{s } ACD,\ BCD, \left\{\begin{array}{ll} AC = BC & \textit{I. Def.}\ 23 \\ CD = CD & \\ \angle ACD = \angle BCD; & \textit{Const} \end{array}\right.$$

$\therefore AD = BD;$ I. 4
that is, $AB$ is bisected at $D$.

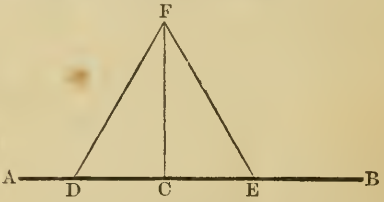
1. Would an isosceles triangle described on $AB$ as base, answer the purpose as well as an equilateral one? If so, why?
2. Prove that $CD$, besides bisecting $AB$, is perpendicular to $AB$.
3. In the figure to I. 1, suppose the two circles to cut at $C$ and $F$; prove that $CF$ bisects $AB$.
4. Hence give (without proof) a simple method of bisecting a given straight line.
5. In the figure to the third deduction, prove that $AB$ and $CF$ bisect each other perpendicularly.
6. Enunciate the preceding deduction as a property of a rhombus.
7. Divide a given straight line into 4 equal parts.
8. Could the number of equal parts into which a straight line may be divided be extended beyond 4? If so, enumerate the numbers.

9. Find a straight line half as long again as a given straight line.
10. Find a straight line equal to half the sum of two given straight lines.
11. Find a straight line equal to half the difference of two given straight lines.
12. If, in the figure to the proposition, $\angle A$ is bisected by $AF$, which meets $BC$ at $F$, prove $BF = BD$, and $AF = CD$.

---

## PROPOSITION 11. PROBLEM.

*To draw a straight line perpendicular to a given straight line from a given point in the same.*



Let $AB$ be the given straight line, and $C$ the given point in it:
*it is required to draw from $C$ a perpendicular to $AB$.*

In $AC$ take any point $D$,
and from $CB$ cut off $CE = CD$. *I.* 3
On $DE$ describe the equilateral $\triangle DEF$, *I.* 1
and join $CF$. $CF$ shall be $\perp AB$.

$$\text{In } \triangle\text{s } DCF,\ ECF, \left\{\begin{array}{ll} DC = EC & \textit{Const.} \\ CF = CF & \\ DF = EF; & \textit{I. Def.}\ 23 \end{array}\right.$$

$\therefore \angle DCF = \angle ECF$; *I.* 8
$\therefore CF$ is $\perp AB$. *I. Def.* 10

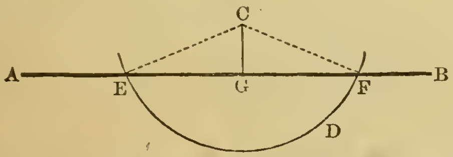
1. Would an isosceles triangle described on $DE$ as base answer the purpose as well as an equilateral one? If so, why?

2. If the given point were situated at either end of the given straight line, what additional construction would be necessary in order to draw a perpendicular?
3. At a given point in a given straight line make an angle equal to half of a right angle.
4. At a given point in a given straight line make an angle equal to one-fourth of a right angle.
5. Construct an isosceles right-angled triangle.
6. Construct a right-angled triangle whose base shall be equal to half the hypotenuse.
7. Find in a given straight line a point which shall be equally distant from two given points. Is this always possible? If not, when is it not?
8. *ABC* is any triangle; *AB* is bisected at *L*, and *AC* at *K*. From *L* there is drawn *LO* perpendicular to *AB*, and from *K*, *KO* perpendicular to *AC*, and these perpendiculars meet at *O*. Prove that *OA*, *OB*, *OC* are all equal.
9. Compare the construction and proof of I. 9 with those of I. 11, and show that the latter proposition is a particular case of the former.

---

## PROPOSITION 12. Problem.

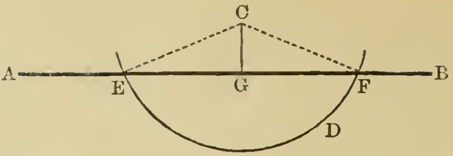
*To draw a straight line perpendicular to a given straight line from a given point without it.*



Let *AB* be the given straight line, and *C* the given point without it:

*it is required to draw from C a perpendicular to AB.*

Take any point *D* on the other side of *AB*;
with centre *C* and radius *CD*, describe the ⊙ *EDF*, cutting *AB*, or *AB* produced, at *E* and *F*.



Bisect $EF$ at $G$; *I.* 10

and join $CG$. $CG$ shall be $\perp AB$.

Join $CE$, $CF$.

$$\text{In } \triangle\text{s } CGE,\ CGF, \left\{\begin{array}{l} EG = FG \qquad \textit{Const.} \\ GC = GC \\ CE = CF; \qquad \textit{I. Def.}\ 16 \end{array}\right.$$

$\therefore \angle CGE = \angle CGF$; *I.* 8

$\therefore CG$ is $\perp AB$. *I. Def.* 10

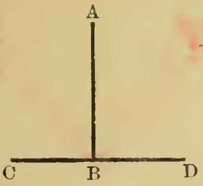
1. Is $CEF$ an equilateral triangle?
2. Prove that $CG$ bisects $\angle ECF$.
3. Instead of bisecting $EF$ at $G$ and joining $CG$, would it answer the purpose equally well to bisect $\angle ECF$ by $CG$?
4. Instead of taking $D$ on the other side of $AB$, would it answer equally well to take $D$ in $AB$ itself?
5. Two points are situated on opposite sides of a given straight line. Find a point in the straight line such that the straight lines joining it to the two given points may make equal angles with the given straight line. Is this always possible?
6. Use the tenth deduction on I. 8 to obtain another method of drawing the perpendicular.

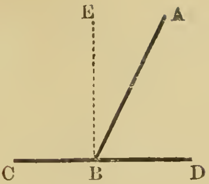
## PROPOSITION 13. THEOREM.

*The angles which one straight line makes with another on one side of it are together equal to two right angles.*

Let $AB$ make with $CD$ on one side of it the $\angle$s $ABC$, $ABD$:

*it is required to prove* $\angle ABC + \angle ABD = 2$ *rt.* $\angle$*s.*

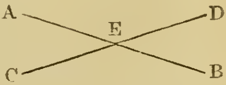




(1) If $\angle ABC = \angle ABD$,
then each of them is a right angle; *I. Def.* 10
$\therefore \angle ABC + \angle ABD = 2$ rt. $\angle$s.

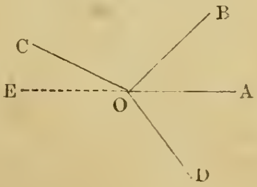
(2) If $\angle ABC$ be not $= \angle ABD$,
from $B$ draw $BE \perp CD$. *I.* 11
Then $\angle$s $EBC$, $EBD$ are 2 rt. $\angle$s. *Const.*
But $\angle ABC + \angle ABD = \angle EBC + \angle EBD$; *I. Ax.* 9
$\therefore \angle ABC + \angle ABD = 2$ rt. $\angle$s. *I. Ax.* 1

Cor. 1.—Hence, if two straight lines cut one another, the four angles which they make at the point where they cut are equal to four right angles.



For $\angle AEC + \angle AED = 2$ rt. $\angle$s, *I.* 13
and $\angle BED + \angle BEC = 2$ rt. $\angle$s. *I.* 13
$\therefore \angle AEC + \angle AED + \angle BED + \angle BEC = 4$ rt. $\angle$s.

Cor. 2.—All the successive angles made by any number of straight lines meeting at one point are together equal to four right angles.



Let $OA$, $OB$, $OC$, $OD$, which meet at $O$, make the successive angles $AOB$, $BOC$, $COD$, $DOA$: *it is required to prove these* $\angle$*s* $= 4$ *rt.* $\angle$*s.*

Produce $AO$ to $E$.

$$\begin{aligned} \text{Then } &\angle AOB + \angle BOC + \angle COD + \angle DOA \\ &= (\angle AOB + \angle BOE) + (\angle EOD + \angle DOA) \\ &= 2 \text{ rt. } \angle\text{s} \qquad\qquad + 2 \text{ rt. } \angle\text{s.} \qquad\qquad I.\ 13 \\ &= 4 \text{ rt. } \angle\text{s.} \end{aligned}$$

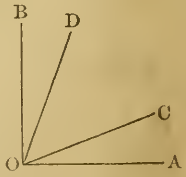
DEF.—Two angles are called **supplementary** when their sum is two right angles; and either angle is called the **supplement** of the other.

Thus, in the figure to the proposition, $\angle ABC$ and $\angle ABD$ are supplementary; $\angle ABC$ is the supplement of $\angle ABD$, and $\angle ABD$ is the supplement of $\angle ABC$.

DEF.—Two angles are called **complementary** when their sum is one right angle; and either angle is called the **complement** of the other.

Thus, in the figure to the proposition, $\angle ABD$ and $\angle ABE$ are complementary; $\angle ABD$ is the complement of $\angle ABE$, and $\angle ABE$ is the complement of $\angle ABD$.

1. In the figure to Cor. 1, name all the angles which are supplementary to $\angle AEC$, to $\angle AED$, to $\angle BED$, to $\angle BEC$.
2. In the figure to Cor. 2, name the angles which are supplementary to $\angle AOB$, $\angle BOE$, $\angle COE$, $\angle EOD$, $\angle AOD$.
3. In the figure to I. 5, name the angles which are supplementary to $\angle ABC$, $\angle ACB$, $\angle DBC$, $\angle ECB$, $\angle BFC$, $\angle CGB$, $\angle ABG$, $\angle ACF$.
4. In the accompanying figure, $\angle AOB$ is right. Name the angles which are complementary to $\angle AOC$, $\angle AOD$, $\angle BOD$, $\angle BOC$.
5. In the same figure, if $\angle AOC = \angle BOD$, prove $\angle AOD = \angle BOC$; and if $\angle AOD = \angle BOC$, prove $\angle AOC = \angle BOD$.



6. In the figure to the proposition, if $\angle$s $ABC$ and $ABD$ be bisected, prove that the bisectors are perpendicular to each other.
7. If the angles at the base of a triangle be equal, the angles on the other side of the base must also be equal.

R 8. If the base of an isosceles triangle be produced both ways, the exterior angles thus formed are equal.

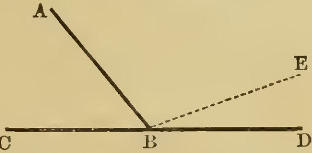
R 9. $ABC$ is a triangle, and the sides $AB$, $AC$ are produced to $D$ and $E$. If $\angle DBC = \angle ECB$, prove $\triangle ABC$ isosceles.

10. $ABC$ is a triangle, and the base $BC$ is produced both ways. If the exterior angles thus formed are equal, prove $\triangle ABC$ isosceles.

---

## PROPOSITION 14. THEOREM.

*If at a point in a straight line, two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.*



At the point $B$ in $AB$, let $BC$ and $BD$, on opposite sides of $AB$, make $\angle ABC + \angle ABD = 2$ rt. $\angle$s:

*it is required to prove $BD$ in the same straight line with $BC$.*

If $BD$ be not in the same straight line with $BC$, produce $CB$ to $E$; *Post.* 2

then $BE$ does not coincide with $BD$.

Now since $CBE$ is a straight line,

$\therefore$ $\angle ABC + \angle ABE = 2$ rt. $\angle$s. *I.* 13

But $\angle ABC + \angle ABD = 2$ rt. $\angle$s; *Hyp.*

$\therefore$ $\angle ABC + \angle ABE = \angle ABC + \angle ABD$. *I. Ax.* 1

Take away from these equals $\angle ABC$, which is common;

$\therefore$ $\angle ABE = \angle ABD$, *I. Ax.* 3

which is impossible;

$\therefore$ $BE$ must coincide with $BD$;

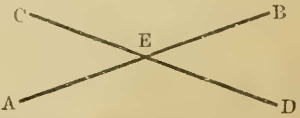
that is, $BD$ is in the same straight line with $BC$.

1. $ABCD$, $EFGH$ are two squares. If they be placed so that $F$ falls on $C$, and $FE$ along $CD$, show that $FG$ will either fall along $CB$, or be in the same straight line with it.
2. If in the straight line $AB$, a point $E$ be taken and two straight lines $EC$, $ED$ be drawn on opposite sides of $AB$, making $\angle AEC = \angle BED$, prove that $EC$ and $ED$ are in the same straight line.
3. If four straight lines, $AE$, $CE$, $BE$, $DE$, meet at a point $E$, so that $\angle AEC = \angle BED$ and $\angle AED = \angle BEC$, then $AE$ and $EB$ are in the same straight line, and also $CE$ and $ED$.
4. $P$ is any point, and $AOB$ a right angle; $PM$ is drawn perpendicular to $OA$ and produced to $Q$, so that $QM = MP$; $PN$ is drawn perpendicular to $OB$ and produced to $R$, so that $RN = NP$. Prove that $Q$, $O$, $R$ lie in the same straight line.
5. If in the enunciation of the proposition the words 'on opposite sides of it' be omitted, is the proposition necessarily true? Draw a figure to illustrate your answer.

---

## PROPOSITION 15. THEOREM.

*If two straight lines cut one another, the vertically opposite angles shall be equal.*



Let $AB$ and $CD$ cut one another at $E$:
*it is required to prove* $\angle AEC = \angle BED$, *and* $\angle BEC = \angle AED$.

Because $CE$ stands on $AB$,

$\therefore$ $\angle AEC + \angle BEC = 2$ rt. $\angle$s. *I.* 13

Because $BE$ stands upon $CD$,

$\therefore$ $\angle BEC + \angle BED = 2$ rt. $\angle$s; *I.* 13

$\therefore$ $\angle AEC + \angle BEC = \angle BEC + \angle BED$. *I. Ax.* 1

Take away from these equals $\angle BEC$, which is common ;

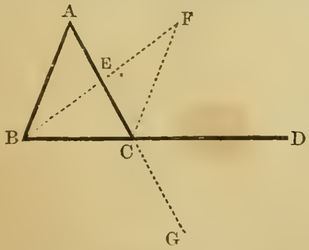
$\therefore \qquad \angle AEC = \angle BED.$ I. Ax. 3

Hence also, $\qquad \angle BEC = \angle AED.$

1. Prove $\angle AEC = \angle BED$, making $\angle AED$ the common angle.
2. " $\angle BEC = \angle AED$, " $\angle AEC$ " "
3. " $\angle BEC = \angle AED$, " $\angle BED$ " "
4. If $\angle AED$ is bisected by $FE$, and $FE$ is produced to $G$, prove that $EG$ bisects $\angle BEC$.
5. If $\angle AED$ is bisected by $FE$, and $\angle BEC$ bisected by $GE$, prove $FE$ and $GE$ in the same straight line.
6. If in a straight line $AB$, a point $E$ be taken, and two straight lines, $EC$, $ED$, be drawn on opposite sides of $AB$, making $\angle AEC = \angle BED$, prove that $EC$ and $ED$ are in the same straight line.
7. $ABC$ is a triangle, $BD$, $CE$ straight lines drawn making equal angles with $BC$, and meeting the opposite sides in $D$ and $E$ and each other in $F$; prove that if $\angle AFE = \angle AFD$, the triangle is isosceles.

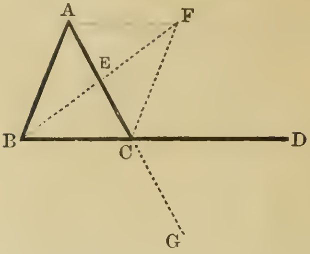
## PROPOSITION 16. THEOREM.

*If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.*



Let $ABC$ be a triangle, and let $BC$ be produced to $D$: *it is required to prove $\angle ACD$ greater than $\angle BAC$, and also greater than $\angle ABC$.*

Bisect $AC$ at $E$; I. 10



join $BE$, and produce it to $F$, making $EF = BE$; *I.* 3
and join $CF$.

$$\text{In } \triangle\text{s } AEB,\ CEF, \left\{ \begin{array}{rr} AE = CE & \textit{Const.} \\ EB = EF & \textit{Const.} \\ \angle\, AEB = \angle\, CEF; & \textit{I.}\ 15 \end{array} \right.$$

$\therefore$ $\angle\, EAB = \angle\, ECF$. *I.* 4

But $\angle\, ACD$ is greater than $\angle\, ECF$; *I. Ax.* 8

$\therefore$ $\angle\, ACD$ is greater than $\angle\, EAB$.

Hence, if $AC$ be produced to $G$,

$\angle\, BCG$ is greater than $\angle\, ABC$.

But $\angle\, ACD = \angle\, BCG$; *I.* 15

$\therefore$ $\angle\, ACD$ is greater than $\angle\, ABC$.

1. Prove $\angle\, A$ less than $AEF$, $BEC$, $ACD$, $BCG$.
2. " $\angle\, F$ " $FCD$, $FCG$, $BEC$, $AEF$.
3. " $\angle\, ABE$ " $AEF$, $BEC$, $ACD$, $BCG$.
4. " $\angle\, CBE$ " $ACD$, $BCG$, $AEB$, $CEF$.
5. " $\angle\, ACB$ " $AEB$, $CEF$.
6. " $\angle\, BEC$ " $ACD$, $BCG$.
7. " $\angle\, BCE$ " $AEB$, $CEF$.
8. " $\angle\, ECF$ " $AEF$, $BEC$.
9. Draw three figures to show that an exterior angle of a triangle may be greater than, equal to, or less than the interior adjacent angle.

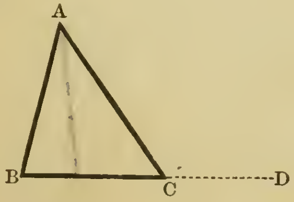
✓ 10. From a point outside a given straight line, there can be drawn to the straight line only one perpendicular.

11. $ABC$ is a triangle whose vertical $\angle\, A$ is bisected by a straight line which meets $BC$ at $D$; prove $\angle\, ADC$ greater than $\angle\, DAC$, and $\angle\, ADB$ greater than $\angle\, BAD$.

12. In the figure to the proposition, if $AF$ be joined, prove: (1) $AF = BC$. (2) Area of $\triangle ABC$ = area of $\triangle BCF$. (3) Area of $\triangle ABF$ = area of $\triangle ACF$.
13. Hence construct on the same base a series of triangles of equal area, whose vertices are equidistant.
14. To a given straight line there cannot be drawn more than two equal straight lines from a given point without it.
15. Any two exterior angles of a triangle are together greater than two right angles.

## PROPOSITION 17. THEOREM.

*The sum of any two angles of a triangle is less than two right angles.*



Let $ABC$ be a triangle:

*it is required to prove the sum of any two of its angles less than 2 rt. ∠s.*

Produce $BC$ to $D$.

Then $\angle ABC$ is less than $\angle ACD$. *I.* 16

∴ $\angle ABC + \angle ACB$ is less than $\angle ACD + \angle ACB$.

But $\angle ACD + \angle ACB$ = 2 rt. ∠s; *I.* 13

∴ $\angle ABC + \angle ACB$ is less than 2 rt. ∠s.

Now $\angle ABC$ and $\angle ACB$ are any two angles of the triangle;

∴ the sum of any two angles of a triangle is less than 2 rt. ∠s.

1. Prove that in any triangle there cannot be two right angles, or two obtuse angles, or one right and one obtuse angle.

2. Prove that in any triangle there must be at least two acute angles.
3. From a point outside a straight line only one perpendicular can be drawn to the straight line.
4. Prove the proposition by joining the vertex to a point inside the base.
5. The angles at the base of an isosceles triangle are both acute.
6. All the angles of an equilateral triangle are acute.
7. If two angles of a triangle be unequal, the smaller of the two must be acute.
8. The three interior angles of a triangle are together less than three right angles.
9. The three exterior angles of a triangle made by producing the sides in succession, are together greater than three right angles.

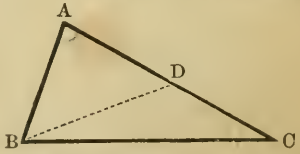
Prove by indirect demonstrations the following theorems :

10. The perpendicular from the right angle of a right-angled triangle on the hypotenuse falls inside the triangle.
11. The perpendicular from the obtuse angle of an obtuse-angled triangle on the opposite side falls inside the triangle.
12. The perpendicular from any of the angles of an acute-angled triangle on the opposite side falls inside the triangle.
13. The perpendicular from any of the acute angles of an obtuse-angled triangle on the opposite side falls outside the triangle.

---

## PROPOSITION 18. THEOREM.

*The greater side of a triangle has the greater angle opposite to it.*



Let $ABC$ be a triangle, having $AC$ greater than $AB$: *it is required to prove* $\angle\ ABC$ *greater than* $\angle\ C$.

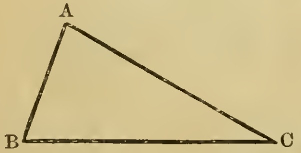
From $AC$ cut off $AD = AB$, *I.* 3
and join $BD$.

Because ∠ $ADB$ is an exterior angle of △ $BCD$,
∴ ∠ $ADB$ is greater than ∠ $C$. *I.* 16
But ∠ $ADB$ = ∠ $ABD$, since $AB = AD$; *I.* 5
∴ ∠ $ABD$ is greater than ∠ $C$.
Much more, then, is ∠ $ABC$ greater than ∠ $C$.

1. If two angles of a triangle be equal, the sides opposite them must also be equal.
2. A scalene triangle has all its angles unequal.
3. If one side of a triangle be less than another side, the angle opposite to it must be acute.
4. $ABCD$ is a quadrilateral whose longest side is $AD$, and whose shortest is $BC$. Prove ∠ $ABC$ greater than ∠ $ADC$, and ∠ $BCD$ greater than ∠ $BAD$.
5. Prove the proposition by producing $AB$ to $D$, so that $AD$ shall be equal to $AC$, and joining $DC$.
6. Prove the proposition from the following construction: Bisect ∠ $A$ by $AD$, which meets $BC$ at $D$; from $AC$ cut off $AE = AB$, and join $DE$.

## PROPOSITION 19. THEOREM.

*The greater angle of a triangle has the greater side opposite to it.*

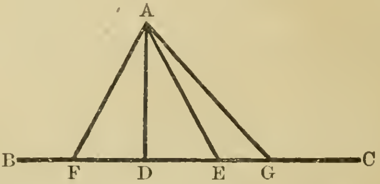


Let $ABC$ be a triangle having ∠ $B$ greater than ∠ $C$:
*it is required to prove $AC$ greater than $AB$.*

If $AC$ be not greater than $AB$,
then $AC$ must be = $AB$, or less than $AB$.
If $AC = AB$, then ∠ $B$ = ∠ $C$. *I.* 5
But it is not;
∴ $AC$ is not = $AB$.

If $AC$ be less than $AB$, then $\angle B$ must be less than $\angle C$. *I.* 18
But it is not;
$\therefore$ $AC$ is not less than $AB$.
Hence $AC$ must be greater than $AB$.

COR.—The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote.



From the given point, $A$, let there be drawn to the given straight line, $BC$, (1) the perpendicular $AD$, (2) $AE$ and $AF$ equally distant from the perpendicular, that is, so that $DE = DF$, (3) $AG$ more remote than $AE$ or $AF$:

*it is required to prove AD the least of these straight lines, and AG greater than AE or AF.*

$$\text{In } \triangle\text{s } ADE,\ ADF, \left\{\begin{array}{l} AD = AD \\ DE = DF \\ \angle ADE = \angle ADF; \end{array}\right. \qquad \begin{array}{r} \\ \textit{Hyp.} \\ \textit{I. Ax.}\ 10 \end{array}$$

$\therefore$ $AE = AF$. *I.* 4
Because $\angle ADE$ is right, $\therefore$ $\angle AED$ is acute; *I.* 17
$\therefore$ $AE$ is greater than $AD$. *I.* 19
Hence also $AF$ is greater than $AD$.

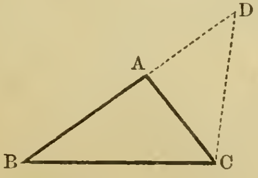
Because $\angle AEG$ is greater than $\angle ADE$, *I.* 16
$\therefore$ $\angle AEG$ is obtuse;
$\therefore$ $\angle AGE$ is acute; *I.* 17
$\therefore$ $AG$ is greater than $AE$. *I.* 19
Hence also $AG$ is greater than $AF$, and than $AD$.

1. The hypotenuse of a right-angled triangle is greater than either of the other sides.
2. A diagonal of a square or of a rectangle is greater than any one of the sides.
3. In an obtuse-angled triangle the side opposite to the obtuse angle is greater than either of the other sides.
4. From $A$, one of the angular points of a square $ABCD$, a straight line is drawn to intersect $BC$ and meet $DC$ produced at $E$; prove that $AE$ is greater than a diagonal of the square. R
5. From a point outside not more than two equal straight lines can be drawn to a given straight line. R
6. The circumference of a circle cannot cut a straight line in more than two points. R
7. $ABC$ is a triangle whose vertical angle $A$ is bisected by a straight line which meets $BC$ at $D$; prove that $AB$ is greater than $BD$, and $AC$ greater than $CD$.

---

## PROPOSITION 20. THEOREM.

*The sum of any two sides of a triangle is greater than the third side.*

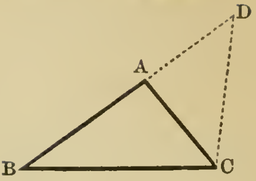


Let $ABC$ be a triangle:

*it is required to prove that the sum of any two of its sides is greater than the third side.*

Produce $BA$ to $D$, making $AD = AC$, *I.* 3
and join $CD$.

Then $\angle ACD = \angle D$, since $AD = AC$. *I.* 5
But $\angle BCD$ is greater than $\angle ACD$;
$\therefore$ $\angle BCD$ is greater than $\angle D$;
$\therefore$ $BD$ is greater than $BC$. *I.* 19



But $BD = BA + AC$;

$\therefore$ $BA + AC$ is greater than $BC$.

Now $BA$ and $AC$ are any two sides;

$\therefore$ the sum of any two sides of a triangle is greater than the third side.

Cor.—The difference of any two sides of a triangle is less than the third side.

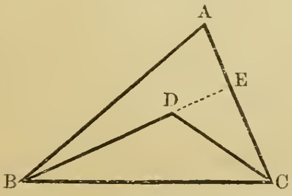
For $BA + AC$ is greater than $BC$. *I.* 20
Taking $AC$ from each of these unequals,
there remains $BA$ greater than $BC - AC$; *I. Ax.* 5
that is, the third side is greater than the difference between the other two.

1. Prove the proposition by producing $CA$ instead of $BA$.
2. " " drawing a perpendicular from the vertex to the base.
3. " " bisecting the vertical angle.
4. In the first figure to I. 7, the sum of $AD$ and $BC$ is greater than the sum of $AC$ and $BD$.
5. A diameter of a circle is greater than any other straight line in the circle which is not a diameter.
6. Any side of a quadrilateral is less than the sum of the other three sides.
7. Any side of a polygon is less than the sum of the other sides.
8. The sum of the distances of any point from the three angles of a triangle is greater than the semi-perimeter of the triangle. Discuss the three cases when the point is inside the triangle, when it is outside, and when it is on a side.
9. The semi-perimeter of a triangle is greater than any one side, and less than any two sides.
10. The sum of the two diagonals of any quadrilateral is greater than the sum of any pair of opposite sides.

11. The perimeter of a quadrilateral is greater than the sum and less than twice the sum of the two diagonals.
12. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines which can be drawn to the four angles from any other point except the intersection of the diagonals.
13. The sum of any two sides of a triangle is greater than twice the median * drawn to the third side, and the excess of this sum over the third side is less than twice the median.
14. The perimeter of a triangle is greater, and the semi-perimeter is less, than the sum of the three medians.

## PROPOSITION 21. THEOREM.

*If from the ends of any side of a triangle there be drawn two straight lines to a point within the triangle, these straight lines shall be together less than the other two sides of the triangle, but shall contain a greater angle.*

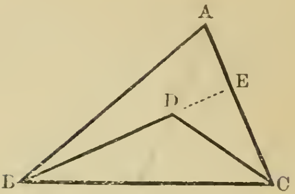


Let $ABC$ be a triangle, and from $B$ and $C$, the ends of $BC$, let $BD$, $CD$ be drawn to any point $D$ within the triangle:

*it is required to prove* (1) *that* $BD + CD$ *is less than* $AB + AC$; (2) *that* $\angle\ BDC$ *is greater than* $\angle\ A$.

---

* DEF.—A *median line*, or a *median*, is a straight line drawn from any vertex of a triangle to the middle point of the opposite side.



Produce $BD$ to meet $AC$ at $E$.

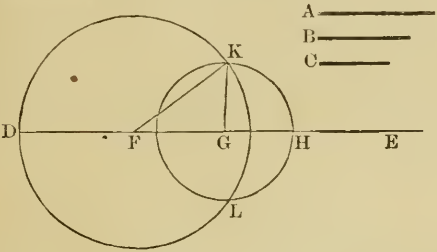
(1) Because $BA + AE$ is greater than $BE$; *I.* 20
add to each of these unequals $EC$;
$\therefore$ $BA + AC$ is greater than $BE + EC$. *I. Ax.* 4
Again, $CE + ED$ is greater than $CD$; *I.* 20
add to each of these unequals $DB$;
$\therefore$ $CE + EB$ is greater than $CD + DB$. *I. Ax.* 4
Much more, then, is $BA + AC$ greater than $CD + DB$.

(2) Because $CED$ is a triangle,
$\therefore$ $\angle BDC$ is greater than $\angle DEC$; *I.* 16
and because $BAE$ is a triangle,
$\therefore$ $\angle DEC$ is greater than $\angle A$; *I.* 16
much more, then, is $\angle BDC$ greater than $\angle A$.

1. Prove the first part of the proposition by producing $CD$ instead of $BD$.
2. Prove the second part of the proposition by joining $AD$ and producing it.
3. In the second figure to I. 7, prove that the perimeter of the triangle $ACB$ is greater than that of $ADB$.
4. Prove the same thing with respect to the third figure to I. 7.
5. If a point be taken inside a triangle and joined to the three vertices, the sum of the three straight lines so drawn shall be less than the perimeter of the triangle.
6. If a triangle and a quadrilateral stand on the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.

## PROPOSITION 22. PROBLEM.

*To make a triangle the sides of which shall be equal to three given straight lines, but any two of these must be greater than the third.*



Let $A$, $B$, $C$ be the three given straight lines, any two of which are greater than the third:
*it is required to make a triangle the sides of which shall be respectively equal to* $A$, $B$, $C$.

Take a straight line $DE$ terminated at $D$, but unlimited towards $E$;
and from it cut off $DF = A$, $FG = B$, $GH = C$. *I.* 3
With centre $F$ and radius $FD$, describe the $\odot$ $DKL$;
with centre $G$ and radius $GH$, describe the $\odot$ $HKL$, cutting the other circle at $K$;
join $KF$, $KG$. $KFG$ is the triangle required.

Because $FK = FD$, being radii of $\odot$ $DKL$, *I. Def.* 16
$\therefore$ $FK = A$.
Because $GK = GH$, being radii of $\odot$ $HKL$, *I. Def.* 16
$\therefore$ $GK = C$.
And $FG$ was made $= B$;
$\therefore$ $\triangle$ $KFG$ has its sides respectively equal to $A$, $B$, $C$.

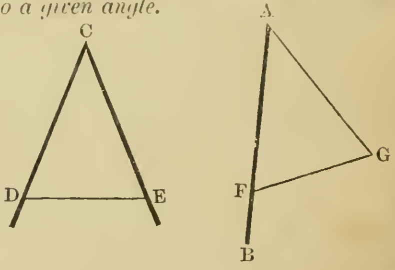
1. Could any other triangle be constructed on the base $FG$ fulfilling the given conditions?

2. If $A$, $B$, $C$ be all equal, which preceding proposition shall we be enabled to solve?
3. Draw a figure showing what will happen when two of the given straight lines are together equal to the third.
4. Draw a figure showing what will happen when two of the given straight lines are together less than the third.
5. Since a quadrilateral can be divided into two triangles by drawing a diagonal, show how to make a quadrilateral whose sides shall be equal to those of a given quadrilateral.
6. Since any rectilineal figure may be decomposed into triangles, show how to make a rectilineal figure whose sides shall be equal to those of a given rectilineal figure.

---

## PROPOSITION 23. PROBLEM.

*At a given point in a given straight line, to make an angle equal to a given angle.*



Let $AB$ be the given straight line, $A$ the given point in it, and $\angle C$ the given angle:
*it is required to make at* $A$ *an angle* $= \angle C$.

In $CD$, $CE$, take any points $D$, $E$, and join $DE$.
Make $\triangle AFG$ such that $AF = CD$, $FG = DE$, $GA = EC$. *I.* 22
$A$ is the required angle.

$$\text{In } \triangle\text{s } AFG,\ CDE, \left\{\begin{array}{ll} AF = CD & \textit{Const.} \\ AG = CE & \textit{Const.} \\ FG = DE; & \textit{Const.} \end{array}\right.$$

$\therefore \angle A = \angle C.$ *I.* 8

1. At a given point in a given straight line, to make an angle equal to the supplement of a given angle.
2. At a given point in a given straight line, to make an angle equal to the complement of a given angle.
3. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.
4. The straight line $OC$ bisects the angle $AOB$; prove that if $OD$ be any other straight line through $O$ without the angle $AOB$, the sum of the angles $DOA$ and $DOB$ is double of the angle $DOC$.
5. The straight line $OC$ bisects the angle $AOB$; prove that if $OD$ be any other straight line through $O$ within the angle $AOB$, the difference of the angles $DOA$ and $DOB$ is double of the angle $DOC$.

Construct an isosceles triangle, having given:

6. The vertical angle and one of the equal sides.
7. The base and one of the angles at the base.

Construct a right-angled triangle, having given:

8. The base and the perpendicular.
9. The base and the acute angle at the base.

Construct a triangle, having given:

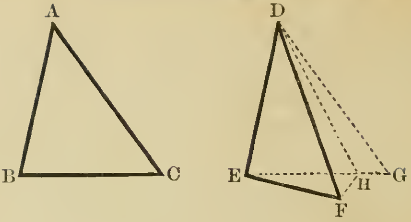
10. The base and the angles at the base.
11. Two sides and the included angle.
12. The base, an angle at the base, and the sum of the other two sides.
13. The base, an angle at the base, and the difference of the other two sides.

## PROPOSITION 24. THEOREM.

*If two triangles have two sides of the one respectively equal to two sides of the other, but the contained angles unequal, the base of the triangle which has the greater contained angle shall be greater than the base of the other.*[*]

---

[*] The proof given in the text is different from Euclid's, which is defective.



Let $ABC$, $DEF$ be two triangles, having $AB = DE$, $AC = DF$, but $\angle BAC$ greater than $\angle EDF$:
*it is required to prove $BC$ greater than $EF$.*

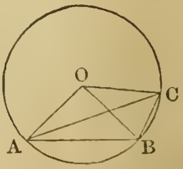
At $D$ make $\angle EDG = \angle BAC$; *I.* 23
cut off $DG = AC$ or $DF$, *I.* 3
and join $EG$.
Bisect $\angle FDG$ by $DH$, meeting $EG$ at $H$; *I.* 9
and, if $F$ does not lie on $EG$, join $FH$.

$$\text{In } \triangle\text{s } ABC, DEG, \left\{\begin{array}{c} BA = ED \\ AC = DG \\ \angle BAC = \angle EDG; \end{array}\right. \qquad \begin{array}{r} \textit{Hyp.} \\ \textit{Const.} \\ \textit{Const.} \end{array}$$

$\therefore BC = EG$. *I.* 4

$$\text{In } \triangle\text{s } FDH, GDH, \left\{\begin{array}{c} FD = GD \\ DH = DH \\ \angle FDH = \angle GDH; \end{array}\right. \qquad \begin{array}{r} \textit{Const.} \\ \\ \textit{Const.} \end{array}$$

$\therefore FH = GH$. *I.* 4
Hence $EH + FH = EH + GH = EG$.
But $EH + FH$ is greater than $EF$; *I.* 20
$\therefore EG$ is greater than $EF$;
$\therefore BC$ is greater than $EF$.



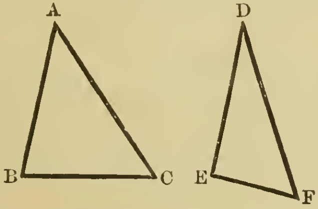
1. $ABC$ is a circle whose centre is $O$. If $\angle AOB$ is greater than $\angle BOC$, prove that $AB$ is greater than $BC$.
2. In the same figure, prove that $AC$ is greater than $AB$ or $BC$.
3. $ABCD$ is a quadrilateral, having $AB = CD$, but $\angle BCD$ greater than $\angle ABC$; prove that $BD$ is greater than $AC$.

4. $ABC$ is an isosceles triangle, having $AB = AC$. $AD$ drawn to the base $BC$ does not bisect $\angle A$; prove that $D$ is at unequal distances from $B$ and $C$.
5. Prove the proposition with the same construction as in the text, but let $\triangle DEG$ fall on the other side of $DE$.

---

## PROPOSITION 25. THEOREM.

*If two triangles have two sides of the one respectively equal to two sides of the other, but their bases unequal, the angle contained by the two sides of the triangle which has the greater base shall be greater than the angle contained by the two sides of the other.*



Let $ABC$, $DEF$ be two triangles, having $AB = DE$, $AC = DF$, but base $BC$ greater than base $EF$:
*it is required to prove $\angle A$ greater than $\angle D$.*

If $\angle A$ be not greater than $\angle D$, it must be either equal to $\angle D$, or less than $\angle D$.

But $\angle A$ is not $= \angle D$, for then base $BC$ would be $=$ base $EF$, *I.* 4
which it is not. *Hyp.*
And $\angle A$ is not less than $\angle D$, for then base $BC$ would be less than base $EF$, *I.* 24
which it is not. *Hyp.*
$\therefore \angle A$ must be greater than $\angle D$.

1. In the figure to the first deduction on I. 24, if $AB$ is greater than $BC$, prove that $\angle AOB$ is greater than $\angle BOC$.

2. $ABCD$ is a quadrilateral, having $AB = CD$, but the diagonal $BD$ greater than the diagonal $AC$; prove that $\angle\ DCB$ is greater than $\angle\ ABC$.
3. $ABCD$ is a quadrilateral, having $AB = CD$, but $\angle\ BCD$ greater than $\angle\ ABC$; prove that $\angle\ DAB$ is greater than $\angle\ ADC$.
4. $ABCD$ is a quadrilateral, having $AB = CD$, but $\angle\ DAB$ greater than $\angle\ ADC$; prove that $\angle\ BCD$ is greater than $\angle\ ABC$.
5. $ABC$ is a triangle, having $AB$ less than $AC$. $D$ is the middle point of $BC$, and $AD$ is joined; prove that $\angle\ ADB$ is acute.
6. $ABC$ is an isosceles triangle, having $AB = AC$. $D$ is any point such that $BD$ is greater than $DC$; prove that $AD$ does not bisect $\angle\ A$.
7. $ABC$ is a triangle, having $AB$ less than $AC$, and $AD$ is the median drawn from $A$; prove that $G$, any point in $AD$, is nearer to $B$ than to $C$.

## PROPOSITION 26. THEOREM.

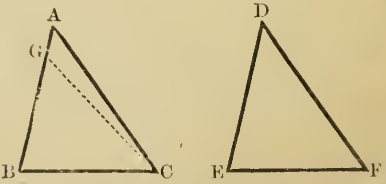
*If two angles and a side in one triangle be respectively equal to two angles and the corresponding side in another triangle, the two triangles shall be equal in every respect; that is,*

(1) *The remaining sides of the one triangle shall be equal to the remaining sides of the other.*

(2) *The third angles shall be equal.*

(3) *The areas of the two triangles shall be equal.*

### CASE 1.



In $\triangle$s $ABC$, $DEF$ let $\angle\ ABC = \angle\ DEF$, $\angle\ ACB = \angle\ DFE$, and $BC = EF$:

*it is required to prove* $AB = DE$, $AC = DF$, $\angle\, A = \angle\, D$, $\triangle\, ABC = \triangle\, DEF$.

If $AB$ be not $= DE$, one of them must be the greater.
Let $AB$ be the greater, and make $BG = DE$; *I.* 3
and join $GC$.

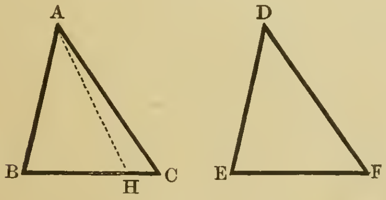
$$\text{In } \triangle\text{s } GBC,\ DEF, \left\{ \begin{array}{ll} GB = DE & \textit{Const.} \\ BC = EF & \textit{Hyp.} \\ \angle\, B = \angle\, E; & \textit{Hyp.} \end{array} \right.$$

$\therefore\ \angle\, GCB = \angle\, DFE$. *I.* 4.
But $\angle\, ACB = \angle\, DFE$; *Hyp.*
$\therefore\ \angle\, GCB = \angle\, ACB$, which is impossible.
Hence $AB$ is not unequal to $DE$, that is, $AB = DE$.

$$\text{Now in } \triangle\text{s } ABC,\ DEF, \left\{ \begin{array}{ll} AB = DE & \textit{Proved} \\ BC = EF & \textit{Hyp.} \\ \angle\, B = \angle\, E; & \textit{Hyp.} \end{array} \right.$$

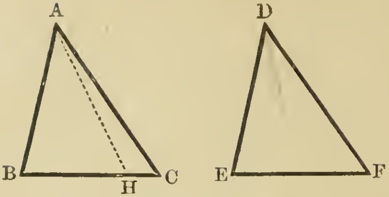
$\therefore\ AC = DF$, $\angle\, A = \angle\, D$, $\triangle\, ABC = \triangle\, DEF$. *I.* 4

### Case 2.



In $\triangle$s $ABC$, $DEF$ let $\angle\, B = \angle\, E$, $\angle\, C = \angle\, F$, and $AB = DE$:
*it is required to prove* $BC = EF$, $AC = DF$, $\angle\, BAC = \angle\, EDF$, $\triangle\, ABC = \triangle\, DEF$.

If $BC$ be not $= EF$, one of them must be the greater.
Let $BC$ be the greater, and make $BH = EF$; *I.* 3
and join $AH$.



$$\text{In } \triangle\text{s } ABH,\ DEF, \begin{cases} AB = DE & \textit{Hyp.} \\ BH = EF & \textit{Const.} \\ \angle B = \angle E; & \textit{Hyp.} \end{cases}$$

$\therefore\ \angle AHB = \angle DFE.$ *I.* 4

But $\angle ACB = \angle DFE$; *Hyp.*

$\therefore\ \angle AHB = \angle ACB$, which is impossible. *I.* 16

Hence $BC$ is not unequal to $EF$, that is, $BC = EF$.

$$\text{Now in } \triangle\text{s } ABC,\ DEF, \begin{cases} AB = DE & \textit{Hyp.} \\ BC = EF & \textit{Proved} \\ \angle B = \angle E; & \textit{Hyp.} \end{cases}$$

$\therefore AC = DF,\ \angle BAC = \angle EDF,\ \triangle ABC = \triangle DEF.$ *I.* 4

1. Prove the first case of the proposition by superposition.
2. The straight line that bisects the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.
3. The straight line drawn from the vertical angle of an isosceles triangle perpendicular to the base, bisects the base and the vertical angle.
4. Any point in the bisector of an angle is equidistant from the arms of the angle.
5. In a given straight line, find a point such that the perpendiculars drawn from it to two other straight lines may be equal.
6. Through a given point, draw a straight line which shall be equidistant from two other given points.
7. Through a given point, draw a straight line which shall form with two given intersecting straight lines an isosceles triangle.

## PROPOSITION A. THEOREM.

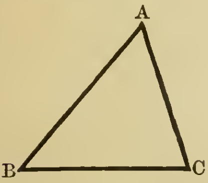
*If two sides of one triangle be respectively equal to two sides of another triangle, and if the angles opposite to one pair of equal sides be equal, the angles opposite the other pair of equal sides shall either be equal or supplementary.*

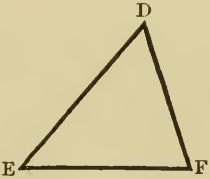
In $\triangle$s $ABC$, $DEF$ let $AB = DE$, $AC = DF$, $\angle B = \angle E$:

*it is required to prove either* $\angle C = \angle F$, *or* $\angle C + \angle F = 2$ *rt.* $\angle$*s.*

$\angle A$ is either $= \angle D$, or not.

CASE 1.—When $\angle A = \angle D$.

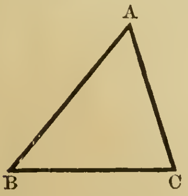


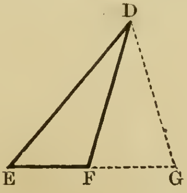


In $\triangle$s $ABC$, $DEF$, $\begin{cases} \angle A = \angle D \\ \angle B = \angle E & \textit{Hyp.} \\ AB = DE; & \textit{Hyp.} \end{cases}$

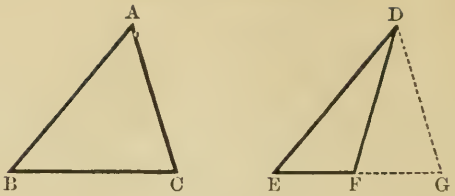
$\therefore$ $\triangle$s $ABC$, $DEF$ are equal in all respects, and $\angle C = \angle F$. *I.* 26

CASE 2.—When $\angle A$ is not $= \angle D$.





At $D$ make $\angle EDG = \angle BAC$; *I.* 23



and let $EF$, produced if necessary, meet $DG$ at $G$.

$$\text{In } \triangle\text{s } ABC,\ DEG, \left\{\begin{array}{l} \angle BAC = \angle EDG \\ \angle ABC = \angle DEG \\ AB = DE; \end{array}\right. \qquad \begin{array}{r} \textit{Const.} \\ \textit{Hyp.} \\ \textit{Hyp.} \end{array}$$

$\therefore\ AC = DG$, and $\angle C = \angle G$. *I.* 26

Now $AC = DF$; *Hyp.*

$\therefore\ DF = DG$;

$\therefore\ \angle DFG = \angle DGF$. *I.* 5

But $\angle DFE$ is supplementary to $\angle DFG$; *I.* 13

$\therefore\ \angle DFE$ is supplementary to $\angle DGF$,

and consequently to $\angle C$.

NOTE.—It often happens that we wish to prove two triangles equal in all respects when we know only that two sides in the one are respectively equal to two sides in the other, and that the angles opposite one pair of equal sides are equal. In such a case, since the angles opposite the other pair of equal sides may either be equal or supplementary, we must endeavour to prove that they cannot be supplementary. To do this, it will be sufficient to know

either (1) that this pair are both acute angles,
or (2) that they are both obtuse angles,
or (3) that one of them is a right angle, since the other must then be a right angle whether it be equal or supplementary to it.

We can tell that this pair of angles must be both acute in certain cases.

(*a*) When the pair of angles given equal are both right angles.
(*b*) " " " " obtuse "
(*c*) " " equal sides opposite the given angles are greater than the other pair of equal sides.

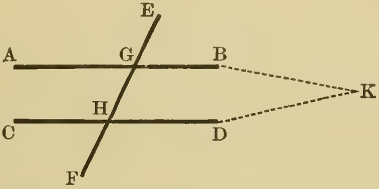
Hence the following important Corollary:

If the hypotenuse and a side of one right-angled triangle be respectively equal to the hypotenuse and a side of another right-angled triangle, the triangles shall be equal in all respects.

---

## PROPOSITION 27. THEOREM.

*If a straight line cutting two other straight lines make the alternate angles equal to one another, the two straight lines shall be parallel.*



Let *EF*, which cuts the two straight lines *AB*, *CD*, make $\angle AGH$ = the alternate $\angle GHD$:
*it is required to prove* $AB \parallel CD$.

If *AB* is not $\parallel$ *CD*, *AB* and *CD* being produced will meet either towards *A* and *C*, or towards *B* and *D*.

Let them be produced, and meet towards *B* and *D* at *K*.

Then *KGH* is a triangle;

$\therefore$ exterior $\angle AGH$ is greater than the interior opposite $\angle GHD$. *I.* 16

But $\angle AGH = \angle GHD$; *Hyp.*

which is impossible.

$\therefore$ *AB* and *CD*, when produced, do not meet towards *B* and *D*.

Hence also, *AB* and *CD*, when produced, do not meet towards *A* and *C*;

$\therefore$ *AB* is $\parallel$ *CD*. *I. Def.* 14

In the figure to I. 16:

1. Prove $AB \parallel CF$.
2. Join $AF$, and prove $AF \parallel BC$.

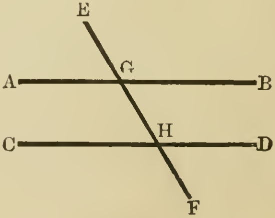
In the figure to I. 28:

3. If $\angle AGE = \angle DHF$, prove $AB \parallel CD$.
4. If $\angle BGE = \angle CHF$, prove $AB \parallel CD$.
5. If $\angle AGE + \angle CHF = 2$ rt. $\angle$s, prove $AB \parallel CD$.
6. If $\angle BGE + \angle DHF = 2$ rt. $\angle$s, prove $AB \parallel CD$.
7. The opposite sides of a square are parallel.
8. The opposite sides of a rhombus are parallel.
9. The quadrilateral whose diagonals bisect each other is a $\parallel^m$

---

## PROPOSITION 28. THEOREM.

*If a straight line cutting two other straight lines make* (1) *an exterior angle equal to the interior opposite angle on the same side of the cutting line, or* (2) *the two interior angles on the same side of the cutting line together equal to two right angles, the two straight lines shall be parallel.*



CASE 1.

Let $EF$, which cuts the two straight lines $AB$, $CD$, make the exterior $\angle EGB$ = the interior opposite $\angle GHD$: *it is required to prove* $AB \parallel CD$.

Because $\angle EGB = \angle GHD$, *Hyp.*

and $\angle EGB = \angle AGH$, being vertically opposite; *I.* 15

$\therefore \angle AGH = \angle GHD$;

and they are alternate angles;

$\therefore AB$ is $\parallel CD$. *I.* 27

### Case 2.

Let *EF*, which cuts the two straight lines *AB*, *CD*, make $\angle BGH + \angle GHD = 2$ rt. $\angle$s:

*it is required to prove* $AB \parallel CD$.

Because $\angle BGH + \angle GHD = 2$ rt. $\angle$s, *Hyp.*

and $\angle AGH + \angle BGH = 2$ rt. $\angle$s; *I.* 13

$\therefore \angle AGH + \angle BGH = \angle BGH + \angle GHD$.

From these equals take $\angle BGH$, which is common;

$\therefore \angle AGH = \angle GHD$; *I. Ax.* 3

and they are alternate angles;

$\therefore AB$ is $\parallel CD$. *I.* 27

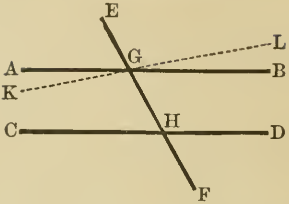
Cor.—Straight lines which are perpendicular to the same straight line are parallel.

1. If $\angle BGE + \angle DHF = 2$ rt. $\angle$s, prove $AB \parallel CD$.
2. If $\angle AGE + \angle CHF = 2$ rt. $\angle$s, prove $AB \parallel CD$.
3. If $\angle AGE = \angle DHF$, prove $AB \parallel CD$.
4. If $\angle BGE = \angle CHF$, prove $AB \parallel CD$.
5. The opposite sides of a square are parallel.
6. *ABCD* is a quadrilateral having $\angle A$ and $\angle B$ supplementary, as well as $\angle B$ and $\angle C$; prove that it is a $\parallel^{m}$.

---

## PROPOSITION 29. Theorem.

*If a straight line cut two parallel straight lines, it shall make* (1) *the alternate angles equal to one another;* (2) *any exterior angle equal to the interior opposite angle on the same side of the cutting line;* (3) *the two interior angles on the same side of the cutting line equal to two right angles.*



Let $EF$ cut the two parallel straight lines $AB$, $CD$: *it is required to prove:*

(1) $\angle AGH =$ *alternate* $\angle GHD$;
(2) *exterior* $\angle EGB =$ *interior opposite* $\angle GHD$;
(3) $\angle BGH + \angle GHD = 2$ *rt.* $\angle$*s.*

(1) If $\angle AGH$ be not $= \angle GHD$, make $\angle KGH = \angle GHD$, *I.* 23
and produce $KG$ to $L$.
Because $\angle KGH =$ alternate $\angle GHD$, *Const.*
$\therefore KL \parallel CD$. *I.* 27
But $AB$ is also $\parallel CD$; *Hyp.*
$\therefore AB$ and $KL$, which cut one another at $G$, are both $\parallel CD$, which is impossible. *I. Ax.* 11
$\therefore \angle AGH$ is not unequal to $\angle GHD$;
$\therefore \angle AGH = \angle GHD$.

(2) Because $\angle AGH = \angle GHD$, *Proved*
and $\angle AGH = \angle EGB$, being vertically opposite; *I.* 15
$\therefore \angle EGB = \angle GHD$.

(3) Because $\angle AGH = \angle GHD$; *Proved*
to each of these equals add $\angle BGH$;
$\therefore \angle AGH + \angle BGH = \angle BGH + \angle GHD$. *I. Ax.* 2
But $\angle AGH + \angle BGH = 2$ rt. $\angle$s; *I.* 13
$\therefore \angle BGH + \angle GHD = 2$ rt. $\angle$s.

Cor.—If a straight line meet two others, and make with them the two interior angles on one side of it together less than two right angles, these two other straight lines will, if produced, meet on that side.

Let $KL$ and $CD$ meet $EF$ and make $\angle KGH + \angle CHG$ less than 2 rt. ∠s:
*it is required to prove that $KG$ and $CH$ will, if produced, meet towards $K$ and $C$.*

If not, $KL$ and $CD$ must either be parallel, or meet towards $L$ and $D$.

(1) $KL$ and $CD$ are not parallel;
for then $\angle KGH + \angle CHG$ would be = 2 rt. ∠s. *I.* 29

(2) $KL$ and $CD$ do not meet towards $L$ and $D$;
for then ∠s $LGH$, $DHG$ would form angles of a triangle,
and would ∴ be together less than 2 rt. ∠s. *I.* 17
Now since the four ∠s $KGH$, $CHG$, $LGH$, $DHG$ are
together = 4 rt. ∠s, *I.* 13
and the first two are less than 2 rt. ∠s; *Hyp.*
∴ the last two must be greater than 2 rt. ∠s.
Hence $KL$ and $CD$ must meet towards $K$ and $C$.

[This Cor. is the converse of I. 17.]

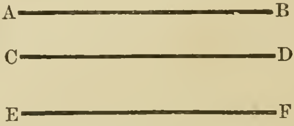
1. In the diagram to I. 28, if $AB$ is ∥ $CD$, prove $\angle AGE = \angle DHF$, and $\angle BGE + \angle DHF = 2$ rt. ∠s.
2. If a straight line be perpendicular to one of two parallels, it is also perpendicular to the other.
3. A straight line drawn parallel to the base of an isosceles triangle, and meeting the sides or the sides produced, forms with them another isosceles triangle.
4. If the arms of one angle be respectively parallel to the arms of another angle, the angles are either equal or supplementary. Distinguish the cases.
5. Is it always true that if two angles be equal, and an arm of the one is parallel to an arm of the other, the other arms must be parallel?

6. If any straight line joining two parallels be bisected, any other straight line drawn through the point of bisection and terminated by the parallels will be bisected at that point.
7. The two straight lines in the last deduction will intercept equal portions of the parallels.
8. If through the vertex of an isosceles triangle a parallel be drawn to the base, it will bisect the exterior vertical angle.
9. If the bisector of the exterior vertical angle of a triangle be parallel to the base, the triangle is isosceles.
10. The diagonals of a ||$^{m}$ bisect each other.
11. Prove that by the following construction $\angle ACB$ is bisected: In $AC$ take any point $D$; draw $DE \perp AC$, and meeting $CB$ at $E$. From $E$ draw $EF \perp DE$ and $= EC$; join $CF$.

---

## PROPOSITION 30. THEOREM.

*Straight lines which are parallel to the same straight line are parallel to one another.*



Let $AB$ and $CD$ be each of them || $EF$:
*it is required to prove* $AB$ || $CD$.

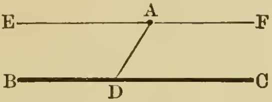
If $AB$ and $CD$ be not parallel, they will meet if produced; and then two straight lines which intersect each other will both be || the same straight line, which is impossible. *I. Ax.* 11

∴ $AB$ is || $CD$.

1. Two ||$^{ms}$ are situated either on the same side or on different sides of a common base. Prove that the sides of the ||$^{ms}$ which are opposite the common base are || each other.
2. Prove the proposition in Euclid's manner by drawing a straight line $GHK$ to cut $AB$, $CD$, and $EF$, and applying I. 29, 27.

## PROPOSITION 31. PROBLEM.

*Through a given point to draw a straight line parallel to a given straight line.*



Let $A$ be the given point, and $BC$ the given straight line: *it is required to draw through $A$ a straight line $\parallel$ $BC$.*

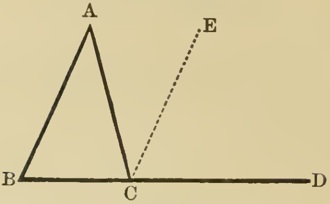
In $BC$ take any point $D$, and join $AD$;
at $A$ make $\angle\ DAE = \angle\ ADC$; *I.* 23
and produce $EA$ to $F$. $EF$ shall be $\parallel$ $BC$.

Because the alternate $\angle$ s $EAD$, $ADC$ are equal,
$\therefore$ $EF$ is $\parallel$ $BC$. *I.* 27

1. Give another construction for the proposition by means of I. 12, 11, and a proof by means of I. 28.
2. Through a given point draw a straight line making with a given straight line an angle equal to a given angle.
3. Through a given point draw a straight line which shall form with two given intersecting straight lines an isosceles triangle.
4. Through a given point draw a straight line such that the part of it intercepted between two parallels may be equal to a given straight line. May there be more than one solution to this problem? Is the problem ever impossible?

## PROPOSITION 32. THEOREM.

*If a side of a triangle be produced, the exterior angle is equal to the sum of the two interior opposite angles, and the sum of the three interior angles is equal to two right angles.*



Let $ABC$ be a triangle having $BC$ produced to $D$:
*it is required to prove* (1) $\angle ACD = \angle A + \angle B$;
(2) $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

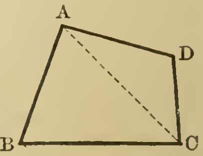
Through $C$ draw $CE \parallel AB$. *I.* 31

(1) Because $AC$ meets the parallels $AB$, $CE$,
$\therefore \angle A =$ alternate $\angle ACE$. *I.* 29
Because $BD$ meets the parallels $AB$, $CE$,
$\therefore$ interior $\angle B =$ exterior $\angle ECD$; *I.* 29
$\therefore \angle A + \angle B = \angle ACE + \angle ECD$,
$= \angle ACD$.

(2) Because $\angle A + \angle B = \angle ACD$; *Proved*
adding $\angle ACB$ to each of these equals,
$\therefore \angle A + \angle B + \angle ACB = \angle ACD + \angle ACB$,
$= 2$ rt. $\angle$s. *I.* 13

Cor. 1.—If two triangles have two angles of the one respectively equal to two angles of the other, they are mutually equiangular.

For the third angles differ from 2 rt. $\angle$s by equal amounts; $\therefore$ the third angles are equal.

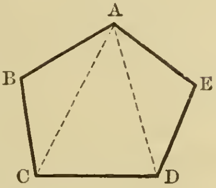


Cor. 2.—The interior angles of a quadrilateral are equal to four right angles.

For the quadrilateral $ABCD$ may be divided into two triangles by joining $AC$; and the six angles of the two $\triangle$s $ABC$, $ACD = 4$ rt. $\angle$s. *I.* 32

But the six angles of the two triangles = the interior angles of the quadrilateral;

∴ the interior angles of the quadrilateral = 4 rt. ∠s.



Cor. 3.—A five-sided figure may be divided into three (that is, $5-2$) triangles by drawing straight lines from one of its angular points. Similarly, a six-sided figure may be divided into four (that is, $6-2$) triangles; and generally a figure of $n$ sides may be divided into $(n-2)$ triangles.

Hence, by a proof like that for the quadrilateral,

the interior ∠s of a five-sided figure = 6 rt. ∠s;
〃 〃 six-sided 〃 = 8 rt. ∠s; and
〃 〃 figure with $n$ sides = $(2n-4)$ rt. ∠s.

1. If an isosceles triangle be right-angled, each of the base angles is half a right angle.
2. If two isosceles triangles have their vertical angles equal, they are mutually equiangular.
3. If one angle of a triangle be equal to the sum of the other two, it must be right.
4. If one angle of a triangle be greater than the sum of the other two, it must be obtuse.
5. If one angle of a triangle be less than the sum of the other two, it must be acute.
6. Divide a right-angled triangle into two isosceles triangles.
7. Hence show that the middle point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.
8. Hence also, devise a method of drawing a perpendicular to given straight line from the end of it without producing the straight line.
9. Each angle of an equilateral triangle is two-thirds of a right angle.
10. Hence show how to trisect* a right angle.

---

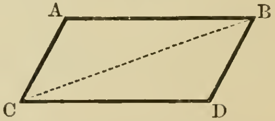
* It is sometimes stated that the problem to trisect *any* angle is beyond the power of Geometry. This is not the case. The problem is beyond the power of Elementary Geometry, which allows the use of only the ruler and the compasses.

11. Prove the second part of the proposition by drawing through $A$ a straight line $DAE \parallel BC$. (The Pythagorean proof.)
12. If any of the angles of an isosceles triangle be two-thirds of a right angle, the triangle must be equilateral.
13. Each of the base angles of an isosceles triangle equals half the exterior vertical angle.
14. If the exterior vertical angle of an isosceles triangle be bisected, the bisector is ∥ the base.
15. Show that the space round a point can be filled up with six equilateral triangles, or four squares, or three regular hexagons.
16. Can a right angle be divided into any other number of equal parts than two or three?
17. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are equiangular to the whole triangle and to one another.
18. Prove the seventh deduction indirectly; and also directly by producing the median to the hypotenuse its own length.
19. If the arms of one angle be respectively perpendicular to the arms of another, the angles are either equal or supplementary.
20. Prove Cor. 3 by taking a point inside the figure and joining it to the angular points.

---

## PROPOSITION 33. THEOREM.

*The straight lines which join the ends of two equal and parallel straight lines towards the same parts, are themselves equal and parallel.*



Let $AB$ and $CD$ be equal and parallel:

*it is required to prove $AC$ and $BD$ equal and parallel.*

Join $BC$.

Because $BC$ meets the parallels $AB$, $CD$,

∴ $\angle ABC =$ alternate $\angle DCB$. *I.* 29

$$\text{In }\triangle\text{s }ABC,\ DCB,\ \left\{\begin{array}{l} AB = DC \\ BC = CB \\ \angle ABC = \angle DCB; \end{array}\right. \qquad \begin{array}{r} \textit{Hyp.} \\ \\ \textit{Proved} \end{array}$$

$\therefore AC = DB$, $\angle ACB = \angle DBC$. *I.* 4

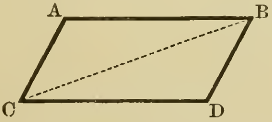
Because $CB$ meets $AC$ and $BD$, and makes the alternate $\angle$s $ACB$, $DBC$ equal; *Proved*

$\therefore AC$ is $\parallel BD$. *I.* 27

1. State a converse of this proposition.
2. If a quadrilateral have one pair of opposite sides equal and parallel, it is a $\parallel^{m}$.
3. What statements may be made about the straight lines which join the ends of two equal and parallel straight lines towards opposite parts?

## PROPOSITION 34. THEOREM.

*A parallelogram has its opposite sides and angles equal, and is bisected by either diagonal.*



Let $ACDB$ be a $\parallel^{m}$ of which $BC$ is a diagonal:

*it is required to prove that the opposite sides and angles of $ACDB$ are equal, and that $\triangle ABC = \triangle DCB$.*

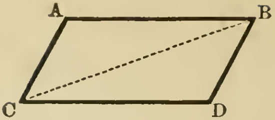
Because $BC$ meets the parallels $AB$, $CD$,

$\therefore \angle ABC =$ alternate $\angle DCB$; *I.* 29

and because $BC$ meets the parallels $AC$, $BD$,

$\therefore \angle ACB =$ alternate $\angle DBC$. *I.* 29

$$\text{In }\triangle\text{s }ABC,\ DCB,\ \left\{\begin{array}{l} \angle ABC = \angle DCB \\ \angle ACB = \angle DBC \\ BC = CB; \end{array}\right. \qquad \begin{array}{r} \textit{Proved} \\ \textit{Proved} \\ \\ \end{array}$$



$\therefore AB = DC, AC = DB, \angle BAC = \angle CDB,$

$\triangle ABC = \triangle DCB.$ *I.* 26

Again because $\angle ABC$ was proved $= \angle DCB,$ *I.* 29

and $\angle DBC$ was proved $= \angle ACB;$ *I.* 29

$\therefore$ the whole $\angle ABD =$ the whole $\angle DCA.$

Cor.—If the arms of one angle be respectively parallel to the arms of another, the angles are either (1) equal or (2) supplementary.

For (1) $\angle BAC$ has been proved $= \angle CDB;$

and (2) if $BA$ be produced to $E$,

$\angle EAC$, which is supplementary to $\angle BAC$, *I.* 13

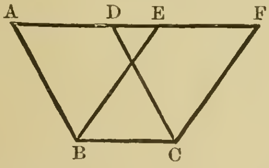
must be supplementary to $\angle CDB$.

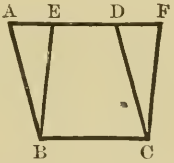
1. If two sides of a ||<sup></sup>m which are not opposite to each other be equal, all the sides are equal.
2. If two angles of a ||m which are not opposite to each other be equal, all the angles are right.
3. If one angle of a ||m be right, all the angles are right.
4. If two ||ms have one angle of the one = one angle of the other, the ||ms are mutually equiangular.
5. If a quadrilateral have its opposite sides equal, it is a ||m.
6. If a quadrilateral have its opposite angles equal, it is a ||m.
7. If the diagonals of a ||m be equal to each other, the ||m is a rectangle.
8. If the diagonals of a ||m bisect the angles through which they pass, the ||m is a rhombus.
9. If the diagonals of a ||m cut each other perpendicularly, the ||m is a rhombus.
10. If the diagonals of a ||m be equal and cut each other perpendicularly, the ||m is a square.
11. Show how to bisect a straight line by means of a pair of parallel rulers.

12. Every straight line drawn through the intersection of the diagonals of a $\|^{m}$, and terminated by a pair of opposite sides, is bisected, and bisects the $\|^{m}$.
13. Bisect a given $\|^{m}$ by a straight line drawn through a given point either within or without the $\|^{m}$.
14. The straight line joining the middle points of any two sides of a triangle is $\|$ the third side and $=$ half of it.
15. If the middle points of the three sides of a triangle be joined with each other, the four triangles thence resulting are equal.
16. Construct a triangle, having given the middle points of its three sides.

## PROPOSITION 35. THEOREM.

*Parallelograms on the same base and between the same parallels are equal in area.*





Let $ABCD$, $EBCF$ be $\|^{ms}$ on the same base $BC$, and between the same parallels $AF$, $BC$:

*it is required to prove* $\|^{m}\ ABCD = \|^{m}\ EBCF$.

Because $AF$ meets the parallels $AB$, $DC$,

$\therefore$ interior $\angle A =$ exterior $\angle FDC$; I. 29

and because $AF$ meets the parallels $EB$, $FC$,

$\therefore$ exterior $\angle AEB =$ interior $\angle F$. I. 29

In $\triangle$s $ABE$, $DCF$, $\left\{\begin{array}{ll} \angle EAB = \angle FDC & \textit{Proved} \\ \angle AEB = \angle DFC & \textit{Proved} \\ AB = DC; & I.\ 34 \end{array}\right.$

$\therefore \triangle ABE = \triangle DCF$. I. 26

Hence quadrilateral $ABCF - \triangle ABE$

$=$ quadrilateral $ABCF - \triangle DCF$;

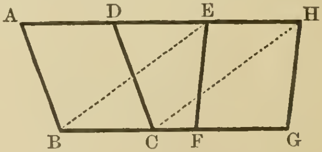
$\therefore \quad \|^{m}\ EBCF = \|^{m}\ ABCD$.

NOTE.—This proposition affords a means of measuring the area of a $\|^{m}$; thence (by I. 34 or 41) the area of a triangle; and thence (by I. 37, Cor.) the area of any rectilineal figure. For the area of any $\|^{m}$ = the area of a rectangle on the same base and between the same parallels; and it is, or ought to be, explained in books on Mensuration, that the area of a rectangle is found by taking the product of its length and breadth. This phrase 'taking the product of its length and breadth,' means that the numbers, whether integral or not, which express the length and breadth in terms of the same linear unit, are to be multiplied together. Hence the method of finding the area of a $\|^{m}$ is to take the product of its base and altitude, the altitude being defined to be the perpendicular drawn to its base from any point in the side opposite.

1. Prove the proposition for the case when the points *D* and *E* coincide.
2. Equal $\|^{ms}$ on the same base and on the same side of it are between the same parallels.
3. If through the vertices of a triangle straight lines be drawn $\|$ the opposite sides, and produced till they meet, the resulting figure will contain three equal $\|^{ms}$.
4. On the same base and between the same parallels as a given $\|^{m}$, construct a rhombus = the $\|^{m}$.
5. Prove the equality of $\triangle$s *ABE* and *DCF* in the proposition by I. 4 (as Euclid does), or by I. 8, instead of by I. 26.

## PROPOSITION 36. THEOREM.

*Parallelograms on equal bases and between the same parallels are equal in area.*



Let *ABCD*, *EFGH* be $\|^{ms}$ on equal bases *BC*, *FG*, and between the same parallels, *AH*, *BG*:
*it is required to prove* $\|^{m}$ *ABCD* = $\|^{m}$ *EFGH*.

Join *BE*, *CH*.

Because *BC* = *FG*, and *FG* = *EH*, *Hyp.*, *I.* 34

∴ *BC* = *EH*.

And because *BC* is ∥ *EH*,

∴ *EB* is ∥ *HC*; *I.* 33

∴ *EBCH* is a ∥$^{m}$. *I. Def.* 33

Now ∥$^{m}$ *ABCD* = ∥$^{m}$ *EBCH*, being on the same base *BC*, and between the same parallels *BC*, *AH*; *I.* 35

and ∥$^{m}$ *EFGH* = ∥$^{m}$ *EBCH*, being on the same base *EH*, and between the same parallels *EH*, *BG*; *I.* 35

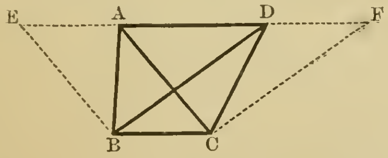
∴ ∥$^{m}$ *ABCD* = ∥$^{m}$ *EFGH*.

1. Prove the proposition by joining *AF*, *DG* instead of *BE*, *CH*.
2. Divide a given ∥$^{m}$ into two equal ∥$^{ms}$.
3. In how many ways may this be done?
4. Of two ∥$^{ms}$ which are between the same parallels, that is the greater which stands on the greater base.
5. State and prove a converse of the last deduction.
6. Equal ∥$^{ms}$ situated between the same parallels have equal bases.

---

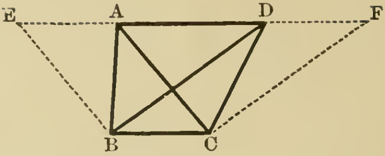
## PROPOSITION 37. THEOREM.

*Triangles on the same base and between the same parallels are equal in area.*



Let *ABC*, *DBC* be triangles on the same base *BC*, and between the same parallels *AD*, *BC*:

*it is required to prove* △ *ABC* = △ *DBC*.



Through $B$ draw $BE \parallel AC$, and through $C$ draw $CF \parallel BD$; *I.* 31

and let them meet $AD$ produced at $E$ and $F$.

Then $EBCA$, $DBCF$ are $\parallel^{ms}$; *I. Def.* 33

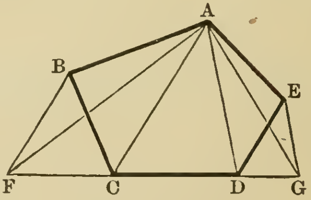
and $\parallel^m EBCA = \parallel^m DBCF$, being on the same base $BC$, and between the same parallels $BC$, $EF$. *I.* 35

But $\triangle ABC =$ half of $\parallel^m EBCA$, *I.* 34

and $\triangle DBC =$ half of $\parallel^m DBCF$; *I.* 34

$\therefore \triangle ABC = \triangle DBC$.

Cor.—Hence any rectilineal figure may be converted into an equivalent triangle.



Let $ABCDE$ be any rectilineal figure:
*it is required to convert it into an equivalent triangle.*

Join $AC$, $AD$;
through $B$ draw $BF \parallel AC$, through $E$ draw $EG \parallel AD$, *I.* 31
and let them meet $CD$ produced at $F$ and $G$.
Join $AF$, $AG$. $AFG$ is the required triangle.

For $\triangle\ AFC = \triangle\ ABC$, and $\triangle\ AGD = \triangle\ AED$; I. 37

$\therefore\ \triangle\ AFC + \triangle\ ACD + \triangle\ AGD = \triangle\ ABC + \triangle\ ACD + \triangle\ AED$.

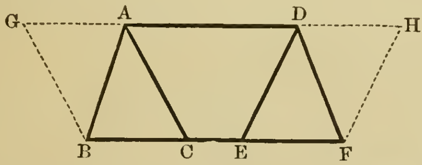
$\therefore\ \triangle\ AFG =$ figure $ABCDE$.

1. $ABC$ is any triangle; $DE$ is drawn $\parallel$ the base $BC$, and meets $AB$, $AC$ at $D$ and $E$; $BE$ and $CD$ are joined. Prove $\triangle\ DBC = \triangle\ EBC$, $\triangle\ BDE = \triangle\ CED$, and $\triangle\ ABE = \triangle\ ACD$.
2. $ABCD$ is a quadrilateral having $AB \parallel CD$; its diagonals $AC$, $BD$ meet at $O$. Prove $\triangle\ AOD = \triangle\ BOC$.
3. In what case would no construction be necessary for the proof of this proposition?
4. Convert a quadrilateral into an equivalent triangle.
5. $ABC$ is any triangle, $D$ a point in $AB$; find a point $E$ in $BC$ produced such that $\triangle\ DBE = \triangle\ ABC$.

---

## PROPOSITION 38. THEOREM.

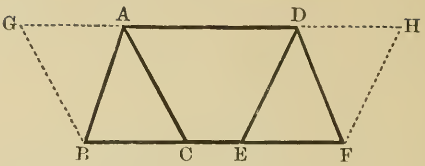
*Triangles on equal bases and between the same parallels are equal in area.*



Let $ABC$, $DEF$ be triangles on equal bases $BC$, $EF$, and between the same parallels $AD$, $BF$:
*it is required to prove* $\triangle\ ABC = \triangle\ DEF$.

Through $B$ draw $BG \parallel AC$, and through $F$ draw $FH \parallel DE$; I. 31
and let them meet $AD$ produced at $G$ and $H$.

Then $GBCA$, $DEFH$ are $\parallel^{ms}$;
and $\parallel^{m}\ GBCA = \parallel^{m}\ DEFH$, being on equal bases $BC$, $EF$,



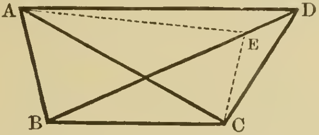
and between the same parallels $BF$, $GH$. *I.* 36
But $\triangle ABC$ = half of $\|^{\text{m}}\ GBCA$, *I.* 34
and $\triangle DEF$ = half of $\|^{\text{m}}\ DEFH$; *I.* 34
$\therefore\ \triangle ABC = \triangle DEF$.

Cor.—The straight line joining any vertex of a triangle to the middle point of the opposite side bisects the triangle. Hence the theorem: If two triangles have two sides of the one respectively equal to two sides of the other and the contained angles supplementary, the triangles are equal in area.

1. Of two triangles which are between the same parallels, that is the greater which stands on the greater base.
2. State and prove a converse of the last deduction.
3. Two triangles are between the same parallels, and the base of the first is double the base of the second; prove the first triangle double the second.
4. The four triangles into which the diagonals divide a $\|^{\text{m}}$ are equal.
5. If one diagonal of a quadrilateral bisects the other diagonal, it also bisects the quadrilateral.
6. $ABCD$ is a $\|^{\text{m}}$; $E$ is any point in $AD$ or $AD$ produced, and $F$ any point in $BC$ or $BC$ produced; $AF$, $DF$, $BE$, $CE$ are joined. Prove $\triangle AFD = \triangle BEC$.
7. $ABC$ is any triangle; $L$ and $K$ are the middle points of $AB$ and $AC$; $BK$ and $CL$ are drawn intersecting at $G$, and $AG$ is joined. Prove $\triangle BGC = \triangle AGC = \triangle AGB$.
8. $ABCD$ is a $\|^{\text{m}}$; $P$ is any point in the diagonal $BD$ or $BD$ produced, and $PA$, $PC$ are joined. Prove $\triangle PAB = \triangle PCB$, and $\triangle PAD = \triangle PCD$.
9. Bisect a triangle by a straight line drawn from a given point in one of the sides.

## PROPOSITION 39. THEOREM.

*Equal triangles on the same side of the same base are between the same parallels.*



Let △s *ABC*, *DBC* on the same side of the same base *BC* be equal, and let *AD* be joined :
*it is required to prove AD* || *BC*.

If *AD* is not || *BC*, through *A* draw *AE* || *BC*, *I.* 31
meeting *BD*, or *BD* produced, at *E*, and join *EC*.

Then △ *ABC* = △ *EBC*. *I.* 37

But △ *ABC* = △ *DBC*; *Hyp.*

∴ △ *EBC* = △ *DBC*;

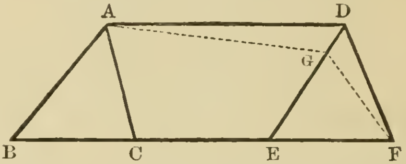
which is impossible, since the one is a part of the other.

∴ *AD* is || *BC*.

1. The straight line joining the middle points of two sides of a triangle is || the third side, and = half of it.
2. Hence prove that the straight line joining the middle point of the hypotenuse of a right-angled triangle to the opposite vertex = half the hypotenuse.
3. The middle points of the sides of any quadrilateral are the vertices of a ||$^{m}$, whose perimeter = the sum of the diagonals of the quadrilateral. When will this ||$^{m}$ be a rectangle, a rhombus, a square?
4. If two equal triangles be on the same base, but on opposite sides of it, the straight line which joins their vertices will be bisected by the base.
5. Use the first deduction to solve I. 31.
6. In the figure to I. 16, prove *AF* || *BC*.
7. If a quadrilateral be bisected by each of its diagonals, it is a ||$^{m}$.
8. Divide a given triangle into four triangles which shall be equal in every respect.

## PROPOSITION 40. THEOREM.

*Equal triangles on the same side of equal bases which are in the same straight line are between the same parallels.*



Let $\triangle$s $ABC$, $DEF$, on the same side of the equal bases $BC$, $EF$, which are in the same straight line $BF$, be equal, and let $AD$ be joined:
*it is required to prove* $AD \parallel BF$.

If $AD$ is not $\parallel BF$, through $A$ draw $AG \parallel BF$, *I.* 31
meeting $DE$, or $DE$ produced, at $G$, and join $GF$.

Then $\triangle ABC = \triangle GEF$. *I.* 38

But $\triangle ABC = \triangle DEF$; *Hyp.*

$\therefore$ $\triangle GEF = \triangle DEF$;

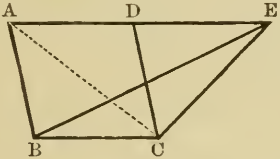
which is impossible, since the one is a part of the other.

$\therefore$ $AD$ is $\parallel BF$.

1. Prove the proposition by joining $AE$ and $AF$.
2. Prove the proposition by joining $DB$ and $DC$.
3. Any number of equal triangles stand on the same side of equal bases. If their bases be in one straight line, their vertices will also be in one straight line.
4. Equal triangles situated between the same parallels have equal bases.
5. Trapeziums on the same base and between the same parallels are equal if the sides opposite the common base are equal.
6. The median from the vertex to the base of a triangle bisects every parallel to the base.
7. Hence devise a method of bisecting a given straight line.

## PROPOSITION 41. THEOREM.

*If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.*



Let the ||$^m$ $ABCD$ and the △ $EBC$ be on the same base $BC$, and between the same parallels $AE$, $BC$:

*it is required to prove* ||$^m$ $ABCD$ = *twice* △ $EBC$.

Join $AC$.

Then △ $ABC$ = △$EBC$. *I.* 37

But ||$^m$ $ABCD$ = twice △ $ABC$; *I.* 34

∴ ||$^m$ $ABCD$ = twice △ $EBC$.

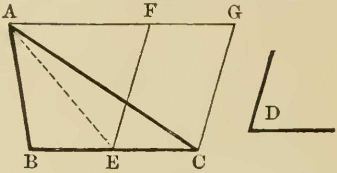
1. Prove the proposition by drawing through $C$ a parallel to $BE$.
2. If a ||$^m$ and a triangle be on equal bases and between the same parallels, the ||$^m$ shall be double of the triangle.
3. A ||$^m$ and a triangle are equal if they are between the same parallels, and the base of the triangle is double that of the ||$^m$.
4. State and prove a converse of the last deduction.
5. If from any point within a ||$^m$ straight lines be drawn to the ends of two opposite sides, the sum of the triangles on these sides shall be equal to half the ||$^m$. Is the theorem true when the point is taken outside? Examine all the cases.
6. $ABCD$ is any quadrilateral, $AC$ and $BD$ its diagonals. A ||$^m$ $EFGH$ is formed by drawing through $A$, $B$, $C$, $D$ parallels to $AC$ and $BD$. Prove $ABCD$ = half of $EFGH$.
7. Hence, show that the area of a quadrilateral = the area of a triangle which has two of its sides equal to the diagonals of the quadrilateral, and the included angle equal to either of

the angles at which the diagonals intersect; and that two quadrilaterals are equal if their diagonals are equal, and also the angles at the intersection of the diagonals.

---

## PROPOSITION 42. Problem.

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.*



Let $ABC$ be the given triangle, and $D$ the given angle: *it is required to describe a* $\|^{m}$ *equal to* $\triangle\ ABC$, *and having one of its angles equal to* $\angle\ D$.

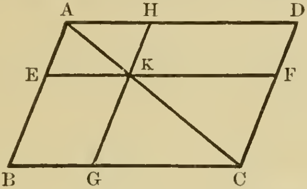
Bisect $BC$ at $E$; *I.* 10
and at $E$ make $\angle\ CEF = \angle\ D$. *I.* 23
Through $A$ draw $AG \parallel BC$; through $C$ draw $CG \parallel EF$. *I.* 31
$FECG$ is the $\|^{m}$ required.

Join $AE$.
The figure $FECG$ is a $\|^{m}$; *I. Def.* 33
and $\|^{m}\ FECG =$ twice $\triangle\ AEC$. *I.* 41
But since $\triangle\ ABE = \triangle\ AEC$, *I.* 38
$\therefore\quad \triangle\ ABC =$ twice $\triangle\ AEC$;
$\therefore\quad \|^{m}\ FECG = \triangle\ ABC$,
and $\angle\ CEF$ was made $= \angle\ D$.

1. Describe a rectangle equal to a given triangle.
2. Describe a triangle that shall be equal to a given $\|^{m}$, and have one of its angles equal to a given angle.
3. On the same base as a $\|^{m}$ construct a right-angled triangle = the $\|^{m}$.
4. Construct a rhombus = a given triangle.

## PROPOSITION 43. THEOREM.

*The complements of the parallelograms which are about a diagonal of any parallelogram are equal.*



Let $ABCD$ be a $\|^{m}$, and $AC$ one of its diagonals; let $EH$, $GF$ be $\|^{ms}$ about $AC$, that is, through which $AC$ passes, and $BK$, $KD$ the other $\|^{ms}$ which fill up the figure $ABCD$, and are therefore called the complements: *it is required to prove complement $BK$ = complement $KD$.*

Because $EH$ is a $\|^{m}$ and $AK$ its diagonal,

$\therefore$ $\triangle AEK = \triangle AHK$. *I.* 34

Similarly $\triangle KGC = \triangle KFC$; *I.* 34

$\therefore \triangle AEK + \triangle KGC = \triangle AHK + \triangle KFC$.

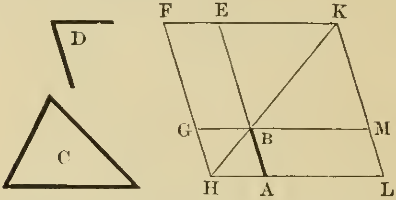
But the whole $\triangle ABC$ = whole $\triangle ADC$; *I.* 34

$\therefore$ the remainder, complement $BK$ = the remainder, complement $KD$.

1. Name the eight $\|^{ms}$ into which $ABCD$ is divided by $EF$ and $GH$, and prove that they are all equiangular to $\|^{m}$ $ABCD$.
2. Prove $\|^{m}$ $AG$ = $\|^{m}$ $ED$, and $\|^{m}$ $BF$ = $\|^{m}$ $DG$.
3. If a point $K$ be taken inside a $\|^{m}$ $ABCD$, and through it parallels be drawn to $AB$ and $BC$, and if $\|^{m}$ $BK$ = $\|^{m}$ $KD$, the diagonal $AC$ passes through $K$. (Converse of I. 43.)
4. Each of the $\|^{ms}$ about a diagonal of a rhombus is itself a rhombus.
5. Each of the $\|^{ms}$ about a diagonal of a square is itself a square.
6. Each of the $\|^{ms}$ about a square's diagonal produced is itself a square.
7. When are the complements of the $\|^{ms}$ about a diagonal of any $\|^{m}$ equal in every respect?

## PROPOSITION 44. PROBLEM.

*On a given straight line to describe a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.*



Let $AB$ be the given straight line, $C$ the given triangle, and $D$ the given angle:
*it is required to describe on* $AB$ *a* $\|^{m} = \triangle C$, *and having an angle* $= \angle D$.

Describe the $\|^{m}$ $BEFG = \triangle C$, and having $\angle EBG = \angle D$; and let it be so placed that $BE$ may be in the same straight line with $AB$. *I.* 42
Through $A$ draw $AH \parallel BG$ or $EF$, *I.* 31
and let it meet $FG$ produced at $H$;
join $HB$.

Because $HF$ meets the parallels $AH$, $EF$,
$\therefore \angle AHF + \angle HFE = 2$ rt. $\angle$s; *I.* 29
$\therefore \angle BHF + \angle HFE$ is less than 2 rt. $\angle$s;
$\therefore HB$, $FE$, if produced, will meet towards $B$, $E$. *I.* 29, *Cor.*
Let them be produced and meet at $K$;
through $K$ draw $KL \parallel EA$ or $FH$, *I.* 31
and produce $HA$, $GB$ to $L$ and $M$.

$ABML$ is the $\|^{m}$ required.

For $FHLK$ is a $\|^{m}$, of which $HK$ is a diagonal,
and $AG$, $ME$ are $\|^{ms}$ about $HK$;

$\therefore$ complement $BL$ = complement $BF$, I. 43

$= \triangle C$.

And $\angle ABM = \angle EBG$, I. 15

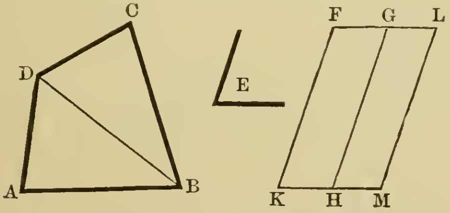
$= \angle D$.

1. On a given straight line describe a rectangle equal to a given triangle.
2. On a given straight line describe a triangle equal to a given $\parallel^{m}$, and having one of its angles equal to a given angle.
3. On a given straight line describe an isosceles triangle equal to a given $\parallel^{m}$.
4. Cut off from a triangle, by a straight line drawn from one of the vertices, a given area.

---

## PROPOSITION 45. PROBLEM.

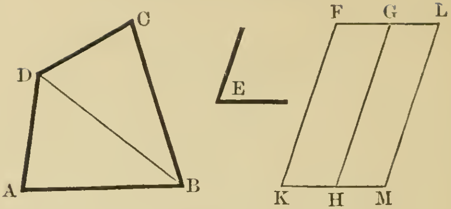
*To describe a parallelogram equal to any given rectilineal figure, and having an angle equal to a given angle.*



Let $ABCD$ be the given rectilineal figure, $E$ the given angle:
*it is required to describe a* $\parallel^{m} = ABCD$, *and having an angle* $= \angle E$.

Join $BD$, and describe the $\parallel^{m}$ $FH = \triangle ABD$, and having $\angle K = \angle E$; I. 42
on $GH$ describe the $\parallel^{m}$ $GM = \triangle BCD$, and having $\angle GHM = \angle E$. I. 44

$FKML$ is the $\parallel^{m}$ required.



Because $\angle K = \angle GHM$, since each $= \angle E$;
to each of these equals add $\angle GHK$;

$\therefore \quad \angle K + \angle GHK = \angle GHM + \angle GHK$.

But $\quad \angle K + \angle GHK = 2$ rt. $\angle$s; *I.* 29

$\therefore \ \angle GHM + \angle GHK = 2$ rt. $\angle$s;

$\therefore$ $KH$ and $HM$ are in the same straight line. *I.* 14

Again, because $FG$ and $GL$ drawn from $G$ are both $\|$ $KM$;

$\therefore$ $FG$ and $GL$ must be in the same straight line. *I. Ax.* 11

Now because $KF$ and $ML$ are both $\|$ $HG$,

$\therefore$ $KF$ is $\|$ $ML$; *I.* 30

and $KM$ is $\|$ $FL$;

$\therefore$ $FKML$ is a $\|^{m}$.

But $\|^{m}$ $FKML = \|^{m}$ $FH$ + $\|^{m}$ $GM$,

$= \triangle ABD + \triangle BCD$, *Const.*

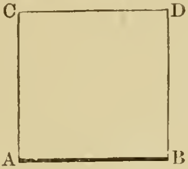
$=$ figure $ABCD$;

and $\angle K = \angle E$. *Const.*

1. Could two $\|^{ms}$ have a common side and together not form one $\|^{m}$? Illustrate by a figure.
2. Describe a rectangle equal to a given rectilineal figure.
3. On a given straight line describe a rectangle equal to a given rectilineal figure.
4. Given one side and the area of a rectangle; find the other side.
5. Describe a $\|^{m}$ equal to a given rectilineal figure, and having an angle equal to a given angle, using I. 37, Cor.
6. Describe a $\|^{m}$ equal to the sum of two given rectilineal figures.
7. Describe a $\|^{m}$ equal to the difference of two given rectilineal figures.

## PROPOSITION 46. PROBLEM.

*On a given straight line to describe a square.*



Let $AB$ be the given straight line:
*it is required to describe a square on AB.*

From $A$ draw $AC \perp AB$ and $= AB$; *I.* 11, 3
through $C$ draw $CD \parallel AB$, *I.* 31
and through $B$ draw $BD \parallel AC$. *I.* 31

$ABDC$ is the square required.

For $ABDC$ is a $\|^{\text{m}}$; *I. Def.* 33
$\therefore$ $AB = CD$ and $AC = BD$. *I.* 34
But $AB = AC$; *Const.*
$\therefore$ the four sides $AB$, $BD$, $DC$, $CA$ are all equal.
Because $AC$ meets the parallels $AB$, $CD$,
$\therefore$ $\angle A + \angle C = 2$ rt. $\angle$s. *I.* 29
But $\angle A$ is right;
$\therefore$ $\angle C$ is also right.
Now $\angle A = \angle D$ and $\angle C = \angle B$; *I.* 34
$\therefore$ the four $\angle$s $A$, $B$, $D$, $C$ are right;
$\therefore$ $ABDC$ is a square. *I. Def.* 32

1. What is redundant in Euclid's definition of a square?
2. If two squares be equal, the sides on which they are described are equal.
3. $ABDC$ is constructed thus: At $A$ and $B$ draw $AC$ and $BD$ $\perp AB$ and $= AB$, and join $CD$. $ABDC$ is a square.
4. $ABDC$ is constructed thus: At $A$ draw $AC \perp AB$ and $= AB$; with $B$ and $C$ as centres, and a radius $= AB$ or $AC$, describe

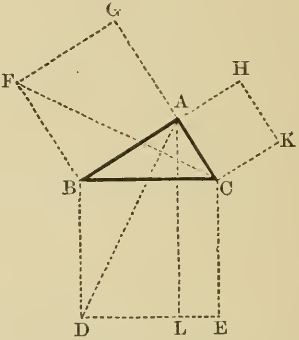
two circles intersecting at *D*; and join *BD*, *DC*. *ABDC* is a square.

5. Describe a square having given a diagonal.

---

## PROPOSITION 47. THEOREM.

*The square described on the hypotenuse of a right-angled triangle is equal to the squares described on the other two sides.*[*]



Let *ABC* be a right-angled triangle, having the right angle *BAC*:

*it is required to prove that the square described on BC = square on BA + square on AC.*

On *AB*, *BC*, *CA* describe the squares *GB*, *BE*, *CH*; *I.* 46

through *A* draw *AL* ∥ *BD* or *CE*; *I.* 31

and join *AD*, *CF*.

Because ∠ *BAC* + ∠ *BAG* = 2 rt. ∠s,

∴ *GA* and *AC* form one straight line. *I.* 14

Similarly, *HA* and *AB* form one straight line.

[*] This theorem is usually attributed to Pythagoras (580—510 B.C.).

Now $\angle DBC = \angle FBA$, each being right.
Add to each $\angle ABC$;
$\therefore\ \angle ABD = \angle FBC$.

$$\text{In } \triangle\text{s } ABD,\ FBC, \left\{\begin{array}{ll} AB = FB & \qquad \textit{I. Def.}\ 32 \\ BD = BC & \qquad \textit{I. Def.}\ 32 \\ \angle ABD = \angle FBC; & \qquad \textit{Proved} \end{array}\right.$$

$\therefore\ \triangle ABD = \triangle FBC$. *I.* 4

But $\|^{\text{m}}\ BL$ = twice $\triangle\ ABD$, being on the same base $BD$, and between the same $\|^{\text{s}}\ BD, AL$; *I.* 41
and square $BG$ = twice $\triangle\ FBC$, being on the same base $BF$, and between the same $\|^{\text{s}}\ BF, CG$; *I.* 41
$\therefore\ \|^{\text{m}}\ BL$ = square $BG$.

Similarly, if $AE$, $BK$ be joined, it may be proved that $\|^{\text{m}}\ CL$ = square $CH$;
$\therefore\ \|^{\text{m}}\ BL + \|^{\text{m}}\ CL$ = square $BG$ + square $CH$,
that is, square on $BC$ = square on $BA$ + square on $AC$.

[It is usual to write this result $BC^2 = BA^2 + AC^2$; but see p. 113.]

Cor.—The difference between the square on the hypotenuse of a right-angled triangle and the square on either of the sides is equal to the square on the other side.

For since $BC^2 = BA^2 + AC^2$,
$\therefore\ BC^2 - BA^2 = AC^2$,
and $BC^2 - AC^2 = BA^2$.

Note.—This proposition is an exceedingly important one, and numerous demonstrations of it have been given by mathematicians, some of them such as easily to afford ocular proof of the equality asserted in the enunciation. With respect to Euclid's method of proof (which is not* that of the discoverer), it may be remarked that he has chosen that position of the squares when they are all exterior to the triangle. The pupil is advised to make the seven other modifications of the figure which result from placing the squares in different positions with respect to the sides of the triangle, and to adapt Euclid's proof thereto. It will be found that $AG$ and $AC$, as well as $AH$ and $AB$, will always be in the same straight line, only, instead of being drawn in opposite directions from $A$ as in the text, they will sometimes be drawn in the same direction; that ∠s $ABD$ and $FBC$ will sometimes be supplementary instead of equal; and that then the equality of △s $ABD$ and $FBC$ will follow, not from I. 4, but from I. 38, Cor.

---

* See Friedlein's *Proclus*, p. 426.

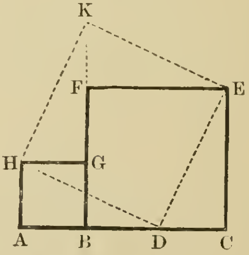
All the different varieties of figure are obtained thus:

Call $X$ the square on the hypotenuse, $Y$ and $Z$ the squares on the other sides. Describe

| | | | |
|---|---|---|---|
| (1) | $X$ outwardly, | $Y$ outwardly, | $Z$ outwardly. |
| (2) | " " | " " | " inwardly. |
| (3) | " " | " inwardly | " outwardly. |
| (4) | " " | " " | " inwardly. |
| (5) | " inwardly, | " outwardly, | " outwardly. |
| (6) | " " | " " | " inwardly. |
| (7) | " " | " inwardly, | " outwardly. |
| (8) | " " | " " | " inwardly. |

The following methods of exhibiting how two squares may be dissected and put together so as to form a third square, are probably the simplest and neatest ocular proofs yet given of this celebrated proposition:

### FIRST METHOD.



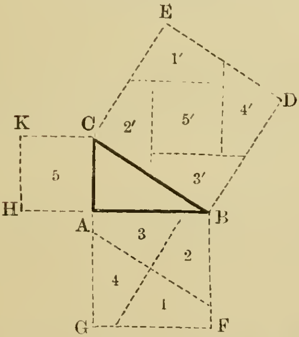
$ABGH$, $BCEF$ are two squares placed side by side, and so that $AB$ and $BC$ form one straight line. Cut off $CD = AB$, and join $ED$, $DH$.

(1) If, round $E$ as a pivot, △ $ECD$ is rotated like the hands of a watch through a right angle, it will occupy the position $EFK$. If, round $H$ as a pivot, △ $HAD$ is rotated in a manner opposite to the hands of a watch through a right angle, it will occupy the position $HGK$. The two squares $ABGH$ and $BCEF$ will then be transformed into the square $DEKH$.

(2) If $\triangle ECD$ be slid along the plane in such a way that $EC$ always remains vertical, and $D$ moves along the line $DH$, it will come to occupy the position $KGH$. If $\triangle HAD$ be slid along the plane in such a way that $HA$ always remains vertical, and $D$ moves along the line $DE$, it will come to occupy the position $KFE$. The two squares $ABGH$ and $BCEF$ will then be transformed into the square $DEKH$.

[This method is substantially that given by Schooten in his *Exercitationes Mathematicæ* (1657), p. 111. The first or rotational way of getting $\triangle$s $ECD$, $HAD$ into their places is given by J. C. Sturm in his *Mathesis Enucleata* (1689), p. 31; the second or translational way is mentioned by De Morgan in the *Quarterly Journal of Mathematics*, vol. i. p. 236.]

### SECOND METHOD.



$ABC$ is a right-angled triangle. $BCED$ is the square on the hypotenuse, $ACKH$ and $ABFG$ are the squares on the other sides.

Find the centre of the square $ABFG$, which may be done by drawing the two diagonals (not shown in the figure), and through it draw two straight lines, one of which is $\parallel BC$, and the other $\perp BC$. The square $ABFG$ is then divided into four quadrilaterals equal in every respect. Through the middle points of the sides of the square $BCED$ draw parallels to $AB$ and $AC$ as in the figure. Then the parts 1, 2, 3, 4, 5 will be found to coincide exactly with $1'$, $2'$, $3'$, $4'$, $5'$.

[This method is due to Henry Perigal, F.R.A.S., and was discovered about 1830. See *The Messenger of Mathematics*, new series, vol. ii. pp. 103–106.]

1. Show how to find a square = the sum of two given squares.
2. " " = " three "
3. " " = the difference of two "
4. " " double of a given square.
5. " " half "
6. " " triple "
7. The square described on a diagonal of a given square is twice the given square.
8. Hence prove that the square on a straight line is four times the square on half the line.
9. The squares described on the two diagonals of a rectangle are together equal to the squares described on the four sides.
10. The squares described on the two diagonals of a rhombus are together equal to the squares described on the four sides.
11. If the hypotenuse and a side of one right-angled triangle be equal to the hypotenuse and a side of another right-angled triangle, the two triangles are equal in every respect.
12. If from the vertex of any triangle a perpendicular be drawn to the base, the difference of the squares on the two sides of the triangle is equal to the difference of the squares on the segments of the base.
13. The square on the side opposite an acute angle of a triangle is less than the squares on the other two sides.
14. The square on the side opposite an obtuse angle of a triangle is greater than the squares on the other two sides.
15. Five times the square on the hypotenuse of a right-angled triangle is equal to four times the sum of the squares on the medians drawn to the other two sides.
16. Three times the square on a side of an equilateral triangle is equal to four times the square on the perpendicular drawn from any vertex to the opposite side.
17. Divide a given straight line into two parts such that the sum of their squares may be equal to a given square. Is this always possible?
18. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.
19. If a straight line be divided into any two parts, the square on the whole line is greater than the sum of the squares on the two parts.

20. The sum of the squares of the distances of any point from two opposite corners of a rectangle is equal to the sum of the squares of its distances from the other two corners.

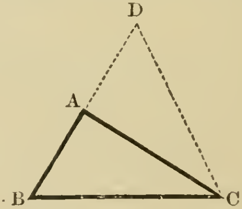
The following deductions refer to the figure of the proposition in the text. They are all, or nearly all, given in an article in *Leybourn's Mathematical Repository*, new series, vol. iii. (1814), Part II. pp. 71–80, by John Bransby, Ipswich.

21. What is the use of proving that $AG$ and $AC$ are in the same straight line, and also $AB$ and $AH$?
22. $AF$ and $AK$ are in the same straight line.
23. $BG$ is $\parallel$ $CH$.
24. Prove $\triangle$s $ABD$, $FBC$ equal by rotating the former round $B$ through a right angle. Similarly, prove $\triangle$s $ACE$, $KCB$ equal.
25. Hence prove $AD \perp FC$, and $AE \perp KB$.
26. $\angle$s $ABC$ and $DBF$ are supplementary, as also are $\angle$s $ACB$ and $ECK$.
27. Hence prove $\triangle$s $FBD$, $KCE = \triangle\ ABC$.
28. $FG$, $KH$, $LA$ all meet at one point $T$.
29. $\triangle$s $AGH$, $THG$, $GAT$, $HTA$ are each $= \triangle\ ABC$.
30. If from $D$ and $E$, perpendiculars $DU$, $EV$ be drawn to $FB$ and $KC$ produced, $\triangle$s $UBD$ and $VEC$ are each $= \triangle\ ABC$. Prove by rotating.
31. $DF^2 + EK^2 = 5\,BC^2$.
32. The squares on the sides of the polygon $DFGHKE = 8\,BC^2$.
33. If from $F$ and $K$ perpendiculars $FM$, $KN$ be drawn to $BC$ produced, and $I$ be the point where $AL$ meets $BC$, $\triangle\ BFM = \triangle\ ABI$, and $\triangle\ CKN = \triangle\ ACI$.
34. $FM + KN = BC$, and $BN = CM = AL$.
35. If $DB$ and $EC$ produced meet $FG$ and $KH$ at $P$ and $Q$, prove by rotating $\triangle\ ABC$ that it = each of the $\triangle$s $FBP$, $KCQ$.
36. If $PQ$ be joined, $BCQP$ is a square.
37. $ABPT$ is a $\parallel^{\text{m}}$, and = rectangle $BL$; $ACQT$ is a $\parallel^{\text{m}}$, and = rectangle $CL$.
38. $ADBT$ is a $\parallel^{\text{m}}$, and = rectangle $BL$; $AECT$ is a $\parallel^{\text{m}}$, and = rectangle $CL$.
39. $DFPU$ and $EKQV$ are $\parallel^{\text{ms}}$, and each $= 4\ \triangle\ ABC$.
40. $ADUH$ and $AEVG$ are $\parallel^{\text{ms}}$, and each $= 2\ \triangle\ ABC$.
41. $BK$ is $\perp CT$, and $CF \perp BT$.
42. Hence prove that $AL$, $BK$, $CF$ meet at one point $O$. (See App. I. 3.)

43. If $BK$ meet $AC$ in $X$, and $CF$ meet $AB$ in $W$, $\triangle$s $BHX$, $CGW$ are each $= \triangle\ ABC$.
44. $AW = AX$.
45. $\triangle\ ACW = \triangle\ BCX$, and $\triangle\ ABX = \triangle\ BCW$.
46. Quadrilateral $AWOX = \triangle\ BOC$.
47. If from $G$ and $H$ perpendiculars $GR$, $HS$ be drawn to $BC$ or $BC$ produced, and if these perpendiculars meet $AB$ and $AC$ in $Y$ and $Z$, prove by rotating $\triangle\ ABC$ that it $= \triangle\ GAY$ or $\triangle\ ZAH$.
48. $DU$ produced passes through $Z$, $EV$ produced through $Y$, $GV$ through $W$, and $HU$ through $X$.
49. If through $A$ a parallel to $BC$ be drawn, meeting $GR$ in $G'$, and $HS$ in $H'$, $\triangle$s $AGG'$, $AZH'$ are $= \triangle\ ABI$, and $\triangle$s $AYG'$ and $AHH' = \triangle\ ACI$.
50. $IR = IS$; $GR + HS = MN$; $FM + GR + HS + KN = 2\ (BC + AI)$; $GR = BS$; $HS = CR$.

## PROPOSITION 48. THEOREM.

*If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by those two sides is a right angle.*



Let $ABC$ be a triangle, and let $BC^2 = BA^2 + AC^2$: *it is required to prove* $\angle\ BAC$ *right.*

From $A$ draw $AD \perp AC$, and $= AB$; *I.* 11, 3
and join $CD$.

Because $AD = AB$; $\therefore AD^2 = AB^2$.
To each of these equals add $AC^2$;
$\therefore AD^2 + AC^2 = AB^2 + AC^2$.

But $AD^2 + AC^2 = CD^2$, *I.* 47

and $AB^2 + AC^2 = BC^2$; *Hyp.*

$\therefore CD^2 = BC^2$;

$\therefore CD = BC$.

$$\text{In } \triangle\text{s } BAC,\ DAC, \begin{cases} BA = DA & \textit{Const.} \\ AC = AC & \\ BC = DC; & \textit{Proved} \end{cases}$$

$\therefore \angle BAC = \angle DAC$, *I.* 8

$= \text{a right angle.}$

1. In the construction it is said, draw $AD \perp AC$. Would it not be simpler, and answer the same purpose, to say, produce $AB$ to $D$. Why?
2. Prove the proposition indirectly by drawing $AD \perp AC$, and on the same side of $AC$ as $AB$, and using I. 7 (Proclus).
3. If the square on one side of a triangle be less than the sum of the squares on the other two sides, the angle opposite that side is acute.
4. If the square on one side of a triangle be greater than the sum of the squares on the other two sides, the angle opposite that side is obtuse.
5. Prove that the triangle whose sides are 3, 4, 5 is right-angled.*
6. Hence derive a method of drawing a perpendicular to a given straight line from a point in it.
7. Show that the following two rules,† due respectively to Pythagoras and Plato, give numbers representing the sides of right-angled triangles, and show also that the two rules are fundamentally the same.

(*a*) Take an odd number for the less side about the right angle. Subtract unity from the square of it, and halve the remainder; this will give the greater side about the right angle. Add unity to the greater side for the hypotenuse.

(*b*) Take an even number for one of the sides about the right angle. From the square of half of this number subtract unity for the other side about the right angle, and to the square of half this number add unity for the hypotenuse.

---

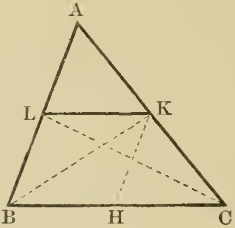
* This is said by Plutarch to have been known to the early Egyptians.

† See Friedlein's *Proclus*, p. 428, and Hultsch's *Heronis . . . reliquiæ*, pp. 56, 57.

# APPENDIX I.

## Proposition 1.

*The straight line joining the middle points of any two sides of a triangle is parallel to the third side and equal to the half of it.*



Let $ABC$ be a triangle, and let $L$, $K$ be the middle points of $AB$, $AC$:
*it is required to prove* $LK \parallel BC$ *and* $=$ *half of* $BC$.

Join $BK$, $CL$.
Because $AL = BL$, $\therefore \triangle BLC =$ half of $\triangle ABC$; I. 38
and because $AK = CK$, $\therefore \triangle BKC =$ half of $\triangle ABC$; I. 38
$\therefore \triangle BLC = \triangle BKC$.
$\therefore LK$ is $\parallel BC$. I. 39
Hence, if $H$ be the middle of $BC$, and $HK$ be joined, $HK$ is $\parallel AB$;
$\therefore BHKL$ is a $\parallel^{m}$; I. Def. 33
$\therefore LK = BH =$ half of $BC$. I. 34

Cor. 1.—Conversely, The straight line drawn through the middle point of one side of a triangle parallel to a second side bisects the third side.*

Cor. 2.—$AB$ is a given straight line, $C$ and $D$ are two points, either on the same side of $AB$ or on opposite sides of $AB$, and such that $AC$ and $BD$ are parallel. If through $E$ the middle point of $AB$, a straight line be drawn $\parallel AC$ or $BD$ to meet $CD$ at $F$, then

---

* The corollaries and converses given in the Appendices should be proved to be true. Many of them are not obvious.

$F$ is the middle point of $CD$, and $EF$ is equal either to half the sum of $AC$ and $BD$, or to half their difference.

### PROPOSITION 2.

*The straight lines drawn perpendicular to the sides of a triangle from the middle points of the sides are concurrent (that is, pass through the same point).*

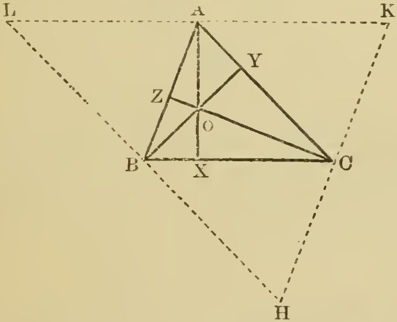
See the figure and demonstration of IV. 5.

If $S$ be joined to $H$, the middle of $BC$, then $SH$ is $\perp BC$. *I.* 8

NOTE.—The point $S$ is called the *circumscribed centre* of $\triangle ABC$.

### PROPOSITION 3.

*The straight lines drawn from the vertices of a triangle perpendicular to the opposite sides are concurrent.*[*]



Let $AX$, $BY$, $CZ$ be the three perpendiculars from $A$, $B$, $C$ on the opposite sides of the $\triangle ABC$;

*it is required to prove $AX$, $BY$, $CZ$ concurrent.*

Through $A$, $B$, $C$ draw $KL$, $LH$, $HK \parallel BC$, $CA$, $AB$. *I.* 31

Then the figures $ABCK$, $ACBL$ are $\parallel^{ms}$; *I. Def.* 33

$\therefore AK = BC = AL$, *I.* 34

that is, $A$ is the middle point of $KL$.

---

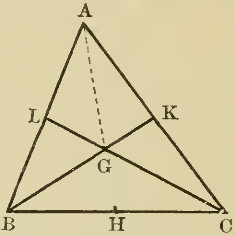
[*] Pappus, VII. 62. The proof here given seems to be due to F. J. Servois: see his *Solutions peu connues de différens problèmes de Géométrie-pratique* (1804), p. 15. It is attributed to Gauss by Dr R. Baltzer.

Hence also, $B$ and $C$ are the middle points of $LH$ and $HK$.
But since $AX$, $BY$, $CZ$ are respectively $\perp$ $BC$, $CA$, $AB$, *Const.*
they must be respectively $\perp$ $KL$, $LH$, $HK$, *I.* 29
and $\therefore$ concurrent. *App. I.* 2

NOTE.—The point $O$ is called the *orthocentre* of the $\triangle$ $ABC$ (an expression due to W. H. Besant), and $\triangle$ $XYZ$, formed by joining the feet of the perpendiculars, is called sometimes the *pedal*, sometimes the *orthocentric*, triangle.

## PROPOSITION 4.

*The medians of a triangle are concurrent.*



Let the medians $BK$, $CL$ of the $\triangle$ $ABC$ meet at $G$:
*it is required to prove that, if $H$ be the middle point of $BC$, the median $AH$ will pass through $G$.*

Join $AG$.
Because $BL = AL$ $\therefore$ $\triangle$ $BLC = \triangle$ $ALC$,
and $\triangle$ $BLG = \triangle$ $ALG$; *I.* 38
$\therefore$ $\triangle$ $BGC = \triangle$ $AGC$, *I. Ax.* 3
$=$ twice $\triangle$ $CKG$; *I.* 38
$\therefore$ $BG =$ twice $GK$, or $BK =$ thrice $GK$,
that is, the median $CL$ cuts $BK$ at its point of trisection remote from $B$.
Hence also, the median $AH$ cuts $BK$ at its point of trisection remote from $B$,
that is, $AH$ passes through G.

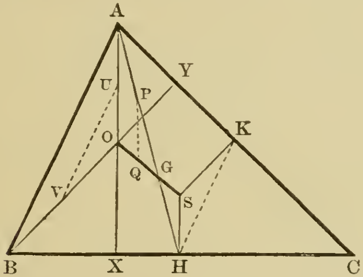
COR.—If the points $H$, $K$, $L$ be joined, the medians of the $\triangle$ $HKL$ are concurrent at $G$.

NOTE.—The point $G$ is called the *centroid* of the $\triangle$ $ABC$ (an expression due to T. S. Davies), and $\triangle HKL$ may be called the *median* triangle. The centroid of a triangle is the same point as that which in Statics is called the centre of gravity of the triangle, and may be found by drawing one median, and trisecting it.

---

### Proposition 5.

*The orthocentre, the centroid, and the circumscribed centre of a triangle are collinear (that is, lie on the same straight line), and the distance between the first two is double of the distance between the last two.*[*]



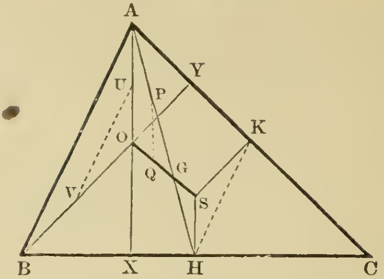
Let $ABC$ be a triangle, $O$ its orthocentre determined by drawing $AX$ and $BY \perp BC$ and $CA$; $S$ its circumscribed centre determined by drawing through $H$ and $K$ the middle points of $BC$ and $CA$, $HS$ and $KS \perp BC$ and $CA$; and $AH$ the median from $A$:
*it is required to prove that if $SO$ be joined, it will cut $AH$ at the centroid.*

Let $SO$ and $AH$ intersect at $G$;
join $P$ and $Q$, the middle points of $GA$, $GO$;
" $U$ " $V$, " " $OA$, $OB$;
and join $HK$.

Because $H$ and $K$ are the middle points of $CB$, $CA$;
$\therefore$ $HK$ is $\parallel$ $AB$ and $=$ half $AB$. *App. I.* 1
Because $U$ and $V$ are the middle points of $OA$, $OB$;
$\therefore$ $UV$ is $\parallel$ $AB$ and $=$ half $AB$, *App. I.* 1
$\therefore$ $HK$ is $\parallel$ $UV$ and $=$ $UV$.

---

[*] First given by Euler in 1765. See *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ*, vol. xi. pp. 13, 114.



Because $SH$ and $OU$ are both $\perp$ $BC$ $\therefore$ $SH$ is $\parallel OU$; *I.* 28, *Cor.*
" $SK$ " $OV$ " $CA$ $\therefore$ $SK$ " $\parallel OV$. *I.* 28, *Cor.*
Hence the $\triangle$s $SHK$, $OUV$ are mutually equiangular, *I.* 34, *Cor.*
and since $HK = UV$ $\therefore$ $SH = OU$ *I.* 26
$= \text{half } AO.$

Again, because $P$ and $Q$ are the middle points of $GA$, $GO$;
$\therefore$ $PQ$ is $\parallel AO$ and $=$ half $AO$; *App. I.* 1
$\therefore$ $PQ$ is $\parallel SH$ and $= SH$.
Hence the $\triangle$s $HGS$, $PGQ$ are equal in all respects; *I.* 29, 26
$\therefore$ $HG = PG =$ half $AG$;
$\therefore$ $G$ is the centroid, *App. I.* 4
and $SG = QG =$ half $OG$.

Cor.—The distance of the circumscribed centre from any side of a triangle is half the distance of the orthocentre from the opposite vertex.

For $SH$ was proved $=$ half $OA$.

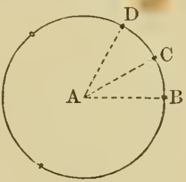
## Loci.

Many of the problems which occur in geometry consist in the finding of points. Now the position of a point—and position is the only property which a point possesses—is determined by certain conditions, and if we know these conditions, we can, in general, find the point which satisfies them. It will be seen that in *plane* geometry *two* conditions suffice to determine a point, provided the conditions be mutually consistent and independent. When only one of the conditions is given, though the point cannot then be determined, yet its position may be so restricted as to enable us to say that wherever the point may be, it must always lie on some one or two lines which we can describe; for example, straight lines or the circumferences of circles. The given condition may, however, be such that the point which satisfies it will lie on a line or lines which we do not as yet know how to describe. Cases where this occurs are considered as not belonging to *elementary* plane geometry.

Def.—The line (or lines) to which a point fulfilling a given condition is restricted, that is, on which alone it can lie, is (or are) called the *locus* of the point. Instead of the phrase 'the locus of a point,' we frequently say 'the locus of points.'

For the complete establishment of a locus, it ought to be proved not only that all the points which are said to constitute the locus fulfil the given condition, but that no other points fulfil it. The latter part of the proof is generally omitted.

Ex. 1. Find the locus of a point having the property (or fulfilling the condition) of being situated at a given distance from a given point.



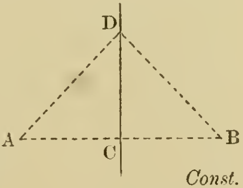
Let $A$ be the given point, and suppose $B$, $C$, $D$, &c. to be points on the locus. Join $AB$, $AC$, $AD$, &c.

Then $AB = AC = AD =$ &c. ; *Hyp.*
and hence $B$, $C$, $D$, &c. must be situated on the ○<sup></sup>ce of a circle whose centre is $A$, and whose radius is the given distance.

Moreover, the distance from $A$ of any point not situated on the ○ce would not be $= AB$, $AC$, $AD$, &c.

This ○ce ∴ is the required locus.

Ex. 2. Find the locus of a point having the property (or fulfilling the condition) of being equidistant from two given points.



Let $A$ and $B$ be the given points.

Join $AB$, and bisect it at $C$; then $C$ is a definite fixed point.

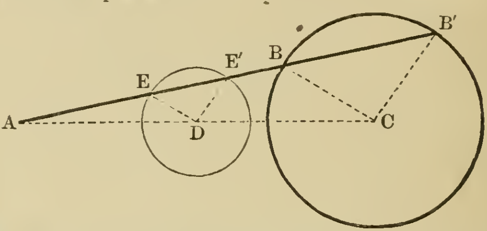
Suppose $D$ to be any point on the locus, and join $DA$, $DB$, $DC$.

Then $DA = DB$; *Hyp.*
and since $DC$ is common, and $AC = BC$, *Const.*
∴ $DC$ is $\perp AB$. *I.* 8, *Def.* 10

Hence, if a set of other points on the locus be taken, and joined to the definite fixed point $C$, a set of perpendiculars to $AB$ will be obtained. The locus therefore consists of all the perpendiculars that can be drawn to $AB$ through the point $C$; that is, $CD$ produced indefinitely either way is the locus.

## Proposition 6.

*Straight lines are drawn from a given fixed point to the circumference of a given fixed circle, and are bisected: find the locus of their middle points.*



Let $A$ be the given fixed point, $C$ the centre of the given fixed circle; let $AB$, one of the straight lines drawn from $A$ to the $\bigcirc^{ce}$, be bisected at $E$:
*it is required to find the locus of E.*

Join $AC$, and bisect it at $D$; *I.* 10
join $DE$ and $CB$.

Because $DE$ joins the middle points of two sides of $\triangle\ ACB$,
$\therefore DE = \frac{1}{2} CB$. *App. I.* 1
But $CB$, being the radius of a fixed circle, is a fixed length;
$\therefore DE$, its half, is also a fixed length.
Again, since $A$ and $C$ are fixed points,
$\therefore AC$ is a fixed straight line;
$\therefore D$, the middle point of $AC$, is a fixed point;
that is, $E$, the middle point of $AB$, is situated at a fixed distance from the fixed point $D$.
But $AB$ was any straight line drawn from $A$ to the $\bigcirc^{ce}$;
$\therefore$ the middle points of all other straight lines drawn from $A$ to the $\bigcirc^{ce}$ must be situated at the same fixed distance from the fixed point $D$;
$\therefore$ the locus of the middle points is the $\bigcirc^{ce}$ of a circle, whose centre is $D$, and whose radius is half the radius of the fixed circle.

From the figure it will be seen that it is immaterial whether $AB$ or $AB'$ is to be considered as the straight line drawn from $A$ to the $\bigcirc^{ce}$. For if $E'$ be the middle point of $AB'$, then $E'D = \frac{1}{2} B'C$, that is $=$ half the radius of the fixed circle;
$\therefore$ the locus of $E'$ is the same $\bigcirc^{ce}$ as before.

[The reader is requested to make figures for the cases when the given point *A* is inside the given circle, and when it is on the ○<sup>ce</sup> of the given circle.]

### INTERSECTION OF LOCI.

Since two conditions determine a point, if we can construct the locus satisfying each condition, the point or points of intersection of the two loci will be the point or points required. A familiar example of this method of determining a point, is the finding of the position of a town on a map by means of parallels of latitude and meridians of longitude. The reader is recommended to apply this method to the solution of I. 1 and 22, and to several of the problems on the construction of triangles.

### DEDUCTIONS.

1. The straight line joining the middle points of the non-parallel sides of a trapezium is ∥ the parallel sides and = half their sum.
2. The straight line joining the middle points of the diagonals of a trapezium is ∥ the parallel sides and = half their difference.
3. The straight line joining the middle points of the non-parallel sides of a trapezium bisects the two diagonals.
4. The middle points of any two opposite sides of a quadrilateral and the middle points of the two diagonals are the vertices of a $\|^{m}$.
5. The straight lines which join the middle points of the opposite sides of a quadrilateral, and the straight line which joins the middle points of the diagonals, are concurrent.
6. If from the three vertices and the centroid of a triangle perpendiculars be drawn to a straight line outside the triangle, the perpendicular from the centroid = one-third of the sum of the other perpendiculars. Examine the cases when the straight line cuts the triangle, and when it passes through the centroid.
7. Find a point in a given straight line such that the sum of its distances from two given points may be the least possible. Examine the two cases, when the two given points are on the same side of the given line, and when they are on different sides.

8. Find a point in a given straight line such that the difference of its distances from two given points may be the greatest possible. Examine the two cases.
9. Of all triangles having only two sides given, that is the greatest in which these sides are perpendicular.
10. The perimeter of an isosceles triangle is less than that of any other triangle of equal area standing on the same base.
11. Of all triangles having the same vertical angle, and the bases of which pass through the same given point, the least is that which has its base bisected by the given point.
12. Of all triangles formed with a given angle which is contained by two sides whose sum is constant, the isosceles triangle has the least perimeter.
13. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the other two sides is constant. Examine the case when the point is in the base produced.
14. The sum of the perpendiculars drawn from any point inside an equilateral triangle to the three sides is constant. Examine the case when the point is outside the triangle.
15. The sum of the perpendiculars from the vertices of a triangle on the opposite sides is greater than the semi-perimeter and less than the perimeter of the triangle.
16. If a perpendicular be drawn from the vertical angle of a triangle to the base, it will divide the vertical angle and the base into parts such that the greater is next the greater side of the triangle.
17. The bisector of the vertical angle of a triangle divides the base into segments such that the greater is next the greater side of the triangle.
18. The median from the vertical angle of a triangle divides the vertical angle into parts such that the greater is next the less side of the triangle.
19. If from the vertex of a triangle there be drawn a perpendicular to the opposite side, a bisector of the vertical angle and a median, the second of these lies in position and magnitude between the other two.
20. The sum of the three angular bisectors of a triangle is greater than the semiperimeter, and less than the perimeter of the triangle.

21. If one side of a triangle be greater than another, the perpendicular on it from the opposite angle is less than the corresponding perpendicular on the other side.
22. If one side of a triangle be greater than another, the median drawn to it is less than the median drawn to the other.
23. If one side of a triangle be greater than another, the bisector of the angle opposite to it is less than the bisector of the angle opposite to the other.
24. The hypotenuse of a right-angled triangle, together with the perpendicular on it from the right angle, is greater than the sum of the other two sides.
25. The sum of the three medians is greater than three-fourths of the perimeter of the triangle.
26. Construct an equilateral triangle, having given the perpendicular from any vertex on the opposite side.

Construct an isosceles triangle, having given :

27. The vertical angle and the perpendicular from it to the base.
28. The perimeter and the perpendicular from the vertex to the base.

Construct a right-angled triangle, having given :

29. The hypotenuse and an acute angle.
30. The hypotenuse and a side.
31. The hypotenuse and the sum of the other sides.
32. The hypotenuse and the difference of the other sides.
33. The perpendicular from the right angle on the hypotenuse and a side.
34. The median, and the perpendicular from the right angle, to the hypotenuse.
35. An acute angle and the sum of the sides about the right angle.
36. An acute angle and the difference of the sides about the right angle.

Construct a triangle, having given :

37. Two sides and an angle opposite to one of them. Examine the cases when the angle is acute, right, and obtuse.
38. One side, an angle adjacent to it, and the sum of the other two sides.
39. One side, an angle adjacent to it, and the difference of the other two sides.
40. One side, the angle opposite to it, and the sum of the other two sides.

41. One side, the angle opposite to it, and the difference of the other two sides.
42. An angle, its bisector, and the perpendicular from the angle on the opposite side.
43. The angles and the sum of two sides.
44. The angles and the difference of two sides.
45. The perimeter and the angles at the base.
46. Two sides and one median.
47. One side and two medians.
48. The three medians.

Construct a square, having given :

49. The sum of a side and a diagonal.
50. The difference of a side and a diagonal.

Construct a rectangle, having given :

51. One side and the angle of intersection of the diagonals.
52. The perimeter and a diagonal.
53. The perimeter and the angle of intersection of the diagonals.
54. The difference of two sides and the angle of intersection of the diagonals.

Construct a $\|^{m}$, having given :

55. The diagonals and a side.
56. The diagonals and their angle of intersection.
57. A side, an angle, and a diagonal.

58. Construct a $\|^{m}$ the area and perimeter of which shall = the area and perimeter of a given triangle.
59. The diagonals of all the $\|^{ms}$ inscribed* in a given $\|^{m}$ intersect one another at the same point.
60. In a given rhombus inscribe a square.
61. In a given right-angled isosceles triangle inscribe a square.
62. In a given square inscribe an equilateral triangle having one of its vertices coinciding with a vertex of the square.
63. $AA'$, $BB'$, $CC'$ are straight lines drawn from the angular points of a triangle through any point $O$ within the triangle, and cutting the opposite sides at $A'$, $B'$, $C'$. $AP$, $BQ$, $CR$ are cut off from $AA'$, $BB'$, $CC'$, and = $OA'$, $OB'$, $OC'$. Prove $\triangle A'B'C' = \triangle PQR$.

---

* One figure is inscribed in another when the vertices of the first figure are on the sides of the second.

64. On $AB$, $AC$, sides of $\triangle ABC$, the $\|^{ms}$ $ABDE$, $ACFG$ are described; $DE$ and $FG$ are produced to meet at $H$, and $AH$ is joined; through $B$ and $C$, $BL$ and $CM$ are drawn $\parallel AH$, and meeting $DE$ and $FG$ at $L$ and $M$. If $LM$ be joined, $BCML$ is a $\|^{m}$, and $= \|^{m}\ BE + \|^{m}\ CG$. (Pappus, IV. 1.)
65. Deduce I. 47 from the preceding deduction.
66. If three concurrent straight lines be respectively perpendicular to the three sides of a triangle, they divide the sides into segments such that the sums of the squares of the alternate segments taken cyclically (that is, going round the triangle) are equal; and conversely.
67. Prove App. I. 2, 3 by the preceding deduction.
68. If from the middle point of the base of a triangle, perpendiculars be drawn to the bisectors of the interior and exterior vertical angles, these perpendiculars will intercept on the sides segments equal to half the sum or half the difference of the sides.
69. In the figure to the preceding deduction, find all the angles which are equal to half the sum or half the difference of the base angles of the triangle.
70. If the straight lines bisecting the angles at the base of a triangle, and terminated by the opposite sides, be equal, the triangle is isosceles. Examine the case when the angles below the base are bisected. [See *Nouvelles Annales de Mathématiques* (1842), pp. 138 and 311; *Lady's and Gentleman's Diary* for 1857, p. 58; for 1859, p. 87; for 1860, p. 84; *London, Edinburgh, and Dublin Philosophical Magazine*, 1852, p. 366, and 1874, p. 354.]

## Loci.

1. The locus of the points situated at a given distance from a given straight line, consists of two straight lines parallel to the given straight line, and on opposite sides of it.
2. The locus of the points situated at a given distance from the $\bigcirc^{ce}$ of a given circle consists of the $\bigcirc^{ces}$ of two circles concentric with the given circle. Examine whether the locus will always consist of two $\bigcirc^{ces}$.

   [The distance of a point from the circumference of a circle is measured on the straight line joining the point to the centre of the circle.]

3. The locus of the points equidistant from two given straight lines which intersect, consists of the two bisectors of the angles made by the given straight lines.
4. What is the locus when the two given straight lines are parallel?
5. The locus of the vertices of all the triangles which have the same base, and one of their sides equal to a given length, consists of the $\bigcirc^{ces}$ of two circles. Determine their centres and the length of their radii.
6. The locus of the vertices of all the triangles which have the same base, and one of the angles at the base equal to a given angle, consists of the sides or the sides produced of a certain rhombus.
7. Find the locus of the centre of a circle which shall pass through a given point, and have its radius equal to a given straight line.
8. Find the locus of the centres of the circles which pass through two given points.
9. Find the locus of the vertices of all the isosceles triangles which stand on a given base.
10. Find the locus of the vertices of all the triangles which have the same base, and the median to that base equal to a given length.
11. Find the locus of the vertices of all the triangles which have the same base and equal altitudes.
12. Find the locus of the vertices of all the triangles which have the same base, and their areas equal.
13. Find the locus of the middle points of all the straight lines drawn from a given point to meet a given straight line.
14. A series of triangles stand on the same base and between the same parallels. Find the locus of the middle points of their sides.
15. A series of $\|^{ms}$ stand on the same base and between the same parallels. Find the locus of the intersection of their diagonals.
16. From any point in the base of a triangle straight lines are drawn parallel to the sides. Find the locus of the intersection of the diagonals of every $\|^{m}$ thus formed.
17. Straight lines are drawn parallel to the base of a triangle, to meet the sides or the sides produced. Find the locus of their middle points.

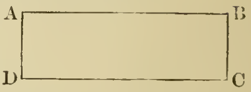
18. Find the locus of the angular point opposite to the hypotenuse of all the right-angled triangles that have the same hypotenuse.
19. A ladder stands upright against a perpendicular wall. The foot of it is gradually drawn outwards till the ladder lies on the ground. Prove that the middle point of the ladder has described part of the $\bigcirc^{ce}$ of a circle.
20. Find the locus of the points at which two equal segments of a straight line subtend equal angles.
21. A straight line of constant length remains always parallel to itself, while one of its extremities describes the $\bigcirc^{ce}$ of a circle. Find the locus of the other extremity.
22. Find the locus of the vertices of all the triangles which have the same base $BC$, and the median from $B$ equal to a given length.
23. The base and the difference of the two sides of a triangle are given; find the locus of the feet of the perpendiculars drawn from the ends of the base to the bisector of the interior vertical angle.
24. The base and the sum of the two sides of a triangle are given; find the locus of the feet of the perpendiculars drawn from the ends of the base to the bisector of the exterior vertical angle.
25. Three sides and a diagonal of a quadrilateral are given: find the locus (1) of the undetermined vertex, (2) of the middle point of the second diagonal, (3) of the middle point of the straight line which joins the middle points of the two diagonals. (*Solutions raisonnées des Problèmes énoncés dans les Éléments de Géométrie de M. A. Amiot*, 7ème ed. p. 124.)

# BOOK II.

## DEFINITIONS.

1. A **rectangle** (or rectangular parallelogram) is said to be contained by any two of its conterminous sides.

Thus the rectangle *ABCD* is said to be contained by *AB* and *BC;* or by *BC* and *CD;* or by *CD* and *DA ;* or by *DA* and *AB.*



The reason of this is, that if the lengths of any two conterminous sides of a rectangle are given, the rectangle can be constructed ; or, what comes to the same thing, that if two conterminous sides of one rectangle are respectively equal to two conterminous sides of another rectangle, the two rectangles are equal in all respects. The truth of the latter statement may be proved by applying the one rectangle to the other.

2. It is oftener the case than not, that the rectangle contained by two straight lines is spoken of when the two straight lines do not actually contain any rectangle. When this is so, the rectangle contained by the two straight lines will signify the rectangle contained by either of them, and a straight line equal to the other, or the rectangle contained by two other straight lines respectively equal to them.

Fig. 1.

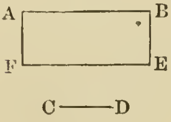


Fig. 2.

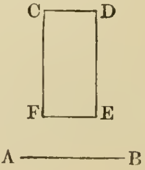
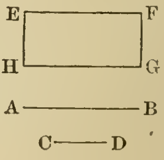


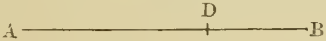
Fig. 3.



Thus $ABEF$ (fig. 1) may be considered the rectangle contained by $AB$ and $CD$, if $BE = CD$; $CDEF$ (fig. 2) may be considered the rectangle contained by $AB$ and $CD$, if $DE = AB$; and $EFGH$ (fig. 3) may be considered the rectangle contained by $AB$ and $CD$, if $EF = AB$ and $FG = CD$.

3. As the rectangle and the square are the figures which the Second Book of Euclid treats of, phrases such as 'the rectangle contained by $AB$ and $AC$,' and 'the square described on $AB$,' will be of constant occurrence. It is usual, therefore, to employ abbreviations for these phrases. The abbreviation which will be made use of in the present text-book* for 'the rectangle contained by $AB$ and $BC$' is $AB \cdot BC$, and for 'the square described on $AB$,' $AB^2$.

4. When a point is taken in a straight line, it is often called a point of **section**, and the distances of this point from the ends of the line are called **segments** of the line.

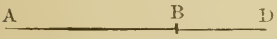


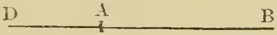
Thus the point of section $D$ divides $AB$ into two segments $AD$ and $BD$.

In this case $AB$ is said to be divided *internally* at $D$, and $AD$ and $BD$ are called *internal segments*.

The given straight line is equal to the sum of its internal segments; for $AB = AD + BD$.

5. When a point is taken in a straight line produced, it is also called a point of **section**, and its distances from the ends of the line are called **segments** of the line.





Thus $D$ is called a point of section of $AB$, and the segments into which it is said to divide $AB$ are $AD$ and $BD$.

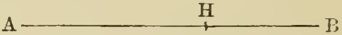
---

* In certain written examinations in England, the only abbreviation allowed for 'the rectangle contained by $AB$ and $BC$' is rect. $AB$, $BC$, and for 'the square described on $AB$,' sq. on $AB$; the pupil, therefore, if preparing for these examinations, should practise himself in the use of such abbreviations.

In this case, $AB$ is said to be divided *externally* at $D$, and $AD$, $BD$ are called *external segments.*

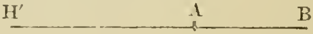
The given straight line is equal to the difference of its external segments; for $AB = AD - BD$, or $BD - AD$.

6. When a straight line is divided into two segments, such that the rectangle contained by the whole line and one of the segments is equal to the square on the other segment, the straight line is said to be divided in **medial section.***



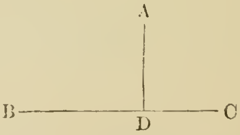
Thus, if $AB$ be divided at $H$ into two segments $AH$ and $BH$, such that $AB \cdot BH = AH^2$, $AB$ is said to be divided in medial section at $H$.

It will be seen that $AB$ is internally divided at $H$; and in general, when a straight line is said to be divided in medial section, it is understood to be internally divided. But the definition need not be restricted to internal division.



Thus, if $AB$ be divided at $H'$ into two segments $AH'$ and $BH'$, such that $AB \cdot BH' = AH'^2$, $AB$ in this case also may be said to be divided in medial section.

7. The **projection** † of a point on a straight line is the foot of the perpendicular drawn from the point to the straight line.



Thus $D$ is the projection of $A$ on the straight line $BC$.

8. The **projection** of one straight line on another straight

---

* The phrase, 'medial section,' seems to be due to Leslie. See his *Elements of Geometry* (1809), p. 66.

† Sometimes the adjective 'orthogonal' is prefixed to the word projection, to distinguish this kind from others.

line is that portion of the second intercepted between perpendiculars drawn to it from the ends of the first.

Fig. 1.

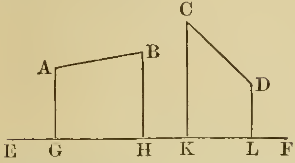
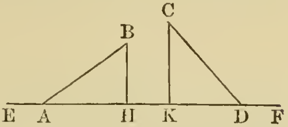


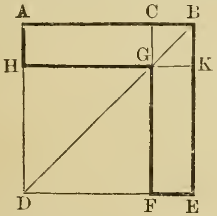
Fig. 2.

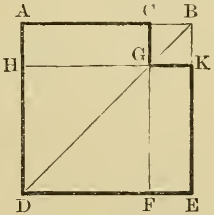


Thus the projections of *AB* and *CD* on *EF* are, in fig. 1, *GH* and *KL* : in fig. 2, *AH* and *KD*.

While the straight line to be projected must be limited in length, the straight line on which it is to be projected must be considered as unlimited.

9. If from a parallelogram there be taken away either of the parallelograms about one of its diagonals, the remaining figure is called a **gnomon.**





Thus if *ADEB* is a $\|^{\text{m}}$, *BD* one of its diagonals, and *HF*, *CK* $\|^{\text{ms}}$ about the diagonal *BD*, the figure which remains when *HF* or *CK* is taken away from *ADEB* is called a gnomon. In the first case, when *HF* is taken away, the gnomon *ABEFGH* (inclosed within thick lines) is usually, for shortness' sake, called *AKF* or *HCE*; in the second case, when *CK* is taken away, the gnomon *ADEKGC* would similarly be called *AFK* or *CHE*.

The word 'gnomon' in Greek means, among other things, a carpenter's square,* which, when the $\|^{\text{m}}$ *ADEB* is a square or a rectangle, the figure $AKF$ resembles. The only gnomons mentioned by Euclid in the second book are parts of squares.

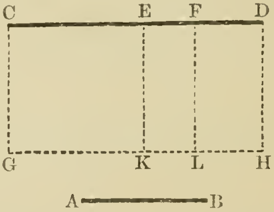
---

* Another less known figure was, from its shape, called by the ancient geometers, 'the shoemaker's knife.' See Pappus, IV. section 14.

The more general definition given by Heron of Alexandria, that a gnomon is any figure which, when added to another figure, produces a figure similar to the original one, will be partly understood after the fourth proposition has been read.

## PROPOSITION 1. THEOREM.

*If there be two straight lines, one of which is divided internally into any number of segments, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several segments of the divided line.*



Let $AB$ and $CD$ be the two straight lines,
and let $CD$ be divided internally into any number of segments $CE$, $EF$, $FD$:
*it is required to prove* $AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$.

From $C$ draw $CG \perp CD$ and $= AB$; *I.* 11, 3
through $G$ draw $GH \parallel CD$,
and through $E$, $F$, $D$ draw $EK$, $FL$, $DH \parallel CG$. *I.* 31

Then $CH = CK + EL + FH$; *I. Ax.* 8
that is, $GC \cdot CD = GC \cdot CE + KE \cdot EF + LF \cdot FD$.
But $GC$, $KE$, $LF$ are each $= AB$; *Const.*, *I.* 34
$\therefore AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$.

ALGEBRAICAL ILLUSTRATION.

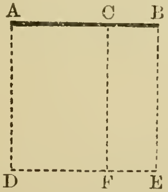
Let $AB = a$, $CD = b$, $CE = c$, $EF = d$, $FD = e$;
then $b = c + d + e$.
Now $AB \cdot CD = ab$,
and $AB \cdot CE + AB \cdot EF + AB \cdot FD = ac + ad + ae$.
But since $b = c + d + e$,
$\therefore ab = ac + ad + ae$;
$\therefore AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$.

1. The rectangle contained by two straight lines is equal to twice the rectangle contained by one of them and half of the other.
2. The rectangle contained by two straight lines is equal to thrice the rectangle contained by one of them and one-third of the other.
3. The rectangle contained by two equal straight lines is equal to the square on either of them.
4. If two straight lines be each of them divided internally into any number of segments, the rectangle contained by the two straight lines is equal to the several rectangles contained by all the segments of the one taken separately with all the segments of the other.

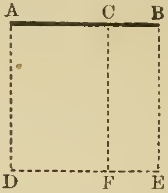
---

## PROPOSITION 2. THEOREM.

*If a straight line be divided internally into any two segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the two segments.*



Let $AB$ be divided internally into any two segments $AC$, $CB$:
*it is required to prove* $AB^2 = AB \cdot AC + AB \cdot CB$.



On $AB$ describe the square $ADEB$, *I.* 46

and through $C$ draw $CF \parallel AD$, meeting $DE$ at $F$. *I.* 31

Then $AE = AF + CE$; *I. Ax.* 8

that is, $AB^2 = DA \cdot AC + EB \cdot CB$.

But $DA$ and $EB$ are each $= AB$;

$\therefore AB^2 = AB \cdot AC + AB \cdot CB$.

### ALGEBRAICAL ILLUSTRATION.

Let $AC = a$, $CB = b$;

then $AB = a + b$.

Now, $AB^2 = (a + b)^2 = a^2 + 2ab + b^2$,

and $AB \cdot AC + AB \cdot CB = (a + b)\,a + (a + b)\,b = a^2 + 2ab + b^2$;

$\therefore AB^2 = AB \cdot AC + AB \cdot CB$.

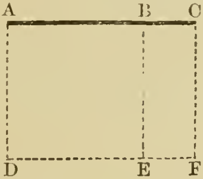
1. Prove this proposition by taking another straight line $= AB$, and using the preceding proposition.
2. If a straight line be divided internally into any three segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the three segments.
3. If a straight line be divided internally into any number of segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the several segments.

Show that the proposition is equivalent to either of the following:

4. The square on the sum of two straight lines is equal to the two rectangles contained by the sum and each of the straight lines.
5. The square on the greater of two straight lines is equal to the rectangle contained by the two straight lines together with the rectangle contained by the greater and the difference between the two.

## PROPOSITION 3. THEOREM.

*If a straight line be divided externally into any two segments, the square on the straight line is equal to the difference of the rectangles contained by the straight line and the two segments.*



Let $AB$ be divided externally into any two segments $AC$, $CB$:

*it is required to prove* $AB^2 = AB \cdot AC - AB \cdot CB$.

On $AB$ describe the square $ADEB$, *I.* 46

and through $C$ draw $CF \parallel AD$, meeting $DE$ produced at $F$. *I.* 31

Then $AE = AF - CE$; *I. Ax.* 8

that is, $AB^2 = DA \cdot AC - EB \cdot CB$.

But $DA$ and $EB$ are each $= AB$;

$\therefore AB^2 = AB \cdot AC - AB \cdot CB$.

NOTE.—The enunciation of this proposition usually given is:

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts together with the square on the aforesaid part.

That is, in reference to the figure,

$$AC \cdot AB = AB^2 + AB \cdot BC,$$

an expression which can be easily derived from that in the text.

### ALGEBRAICAL ILLUSTRATION.

Let $AC = a$, $CB = b$;

then $AB = a - b$.

Now, $AB^2 = (a - b)^2 = a^2 - 2ab + b^2$,

and $AB \cdot AC - AB \cdot CB = (a - b)\,a - (a - b)\,b = a^2 - 2ab + b^2$;
$\therefore AB^2 = AB \cdot AC - AB \cdot CB$.

1. Prove this proposition by taking another straight line $= AB$, and using the first proposition.

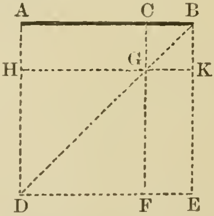
   Show that the proposition is equivalent to either of the following:

2. The rectangle contained by the sum of two straight lines and one of them is equal to the square on that one together with the rectangle contained by the two straight lines.
3. The rectangle contained by two straight lines is equal to the square on the less together with the rectangle contained by the less and the difference of the two straight lines.

---

## PROPOSITION 4. THEOREM.

*If a straight line be divided internally into any two segments, the square on the straight line is equal to the squares on the two segments increased by twice the rectangle contained by the segments.*



Let $AB$ be divided internally into any two segments $AC$, $CB$:

*it is required to prove* $AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.

On $AB$ describe the square $ADEB$, and join $BD$. *I.* 46

Through $C$ draw $CF \parallel AD$, meeting $DB$ at $G$;
and through $G$ draw $HK \parallel AB$, meeting $DA$ and $EB$ at $H$ and $K$. *I.* 31

Because $CG \parallel AD$, $\therefore \angle CGB = \angle ADB$; *I.* 29
and because $AD = AB$, $\therefore \angle ADB = \angle ABD$; *I.* 5

$\therefore \quad \angle\, CGB = \angle\, ABD,$

$\qquad\qquad = \angle\, CBG;$

$\therefore \qquad CB = CG.$ *I.* 6

Hence the $\|^{\text{m}}$ $CK$, having two adjacent sides equal, has all its sides equal. *I.* 34

But the $\|^{\text{m}}$ $CK$ has one of its angles, $KBC$, right, since $\angle\, KBC$ is the same as $\angle\, ABE$;

$\therefore$ it has all its angles right; *I.* 34

$\therefore$ the $\|^{\text{m}}$ $CK$ is a square, and $= CB^2$. *I. Def.* 32

Similarly, the $\|^{\text{m}}$ $HF$ is a square, and $= HG^2 = AC^2$.

Again, the $\|^{\text{m}}$ $AG = AC \cdot CG = AC \cdot CB$;

$\therefore \qquad GE = AC \cdot CB;$ *I.* 43

$\therefore \qquad AG + GE = 2\, AC \cdot CB.$

Now $\qquad AB^2 = ADEB,$

$\qquad\qquad = HF + CK + AG + GE,$ *I. Ax.* 8

$\qquad\qquad = AC^2 + CB^2 + 2\, AC \cdot CB.$

Cor. 1.—The square on the sum of two straight lines is equal to the sum of the squares on the two straight lines, increased by twice the rectangle contained by the two straight lines.

For if $AC$ and $CB$ be the two straight lines,

then their sum $= AC + CB = AB.$

Now since $AB^2 = AC^2 + CB^2 + 2\, AC \cdot CB,$ *II.* 4

$\therefore (AC + CB)^2 = AC^2 + CB^2 + 2\, AC \cdot CB.$

Cor. 2.—The $\|^{\text{ms}}$ about a diagonal of a square are themselves squares.

[It is recommended that II. 7 be read immediately after II. 4.]

OTHERWISE:

$AB^2 = \qquad AB \cdot AC \qquad + \qquad AB \cdot BC,$ *II.* 2

$\quad = (AC \cdot AC + BC \cdot AC) + (AC \cdot BC + BC \cdot BC),$ *II.* 3

$\quad = AC^2 + BC^2 + 2\, AC \cdot BC.$

### ALGEBRAICAL ILLUSTRATION.

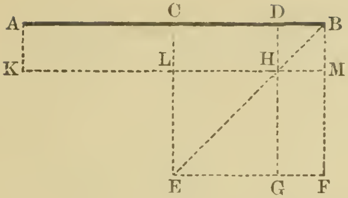
Let $AC = a$, $CB = b$;
then $AB = a + b$.
Now $AB^2 = (a + b)^2 = a^2 + 2ab + b^2$,
and $AC^2 + CB^2 + 2\,AC \cdot CB = a^2 + b^2 + 2ab$;
$\therefore\ AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.

1. Name the two figures which form the sum of the squares on $AC$ and $CB$.
2. Name the figure which is the square on the sum of $AC$ and $CB$.
3. Name the figure which is the difference of the squares on $AB$ and $AC$.
4. Name the figure which is the difference of the squares on $AB$ and $BC$.
5. Name the figure which is the square on the difference of $AB$ and $AC$.
6. Name the figure which is the square on the difference of $AB$ and $BC$.
7. By how much does the square on the sum of $AC$ and $CB$ exceed the sum of the squares on $AC$ and $CB$?
8. Show that the proposition may be enunciated: The square on the sum of two straight lines is greater than the sum of the squares on the two straight lines by twice the rectangle contained by the two straight lines.
9. The square on any straight line is equal to four times the square on half of the line.
10. If a straight line be divided internally into any three segments, the square on the whole line is equal to the squares on the three segments, together with twice the rectangles contained by every two of the segments.
11. Illustrate the preceding deduction algebraically.

---

## PROPOSITION 5. THEOREM.

*If a straight line be divided into two equal, and also internally into two unequal segments, the rectangle contained by the unequal segments is equal to the difference between the square on half the line and the square on the line between the points of section.*



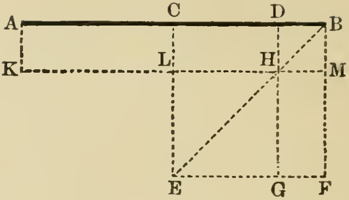
Let $AB$ be divided into two equal segments $AC$, $CB$, and also internally into two unequal segments $AD$, $DB$: *it is required to prove* $AD \cdot DB = CB^2 - CD^2$.

On $CB$ describe the square $CEFB$, and join $BE$. *I.* 46
Through $D$ draw $DHG \parallel CE$, meeting $EB$ and $EF$ at $H$ and $G$;
through $H$ draw $MHLK \parallel AB$, meeting $FB$ and $EC$ at $M$ and $L$;
and through $A$ draw $AK \parallel CL$. *I.* 31

Then $AD \cdot DB = AD \cdot DH$, *II.* 4. *Cor.* 2
$= AH$,
$= AL + CH$, *I. Ax.* 8
$= CM + HF$, *I.* 36, 43
$=$ gnomon $CMG$. *I. Ax.* 8
But $CB^2 - CD^2 = CB^2 - LH^2$, *I.* 34
$= CEFB - LEGH$,
$=$ gnomon $CMG$. *I. Ax.* 8
$\therefore AD \cdot DB = CB^2 - CD^2$.

Cor.—The difference of the squares on two straight lines is equal to the rectangle contained by the sum and the difference of the two straight lines.

Let $AC$ and $CD$ be the two straight lines:
*it is required to prove*
$AC^2 - CD^2 = (AC + CD) \cdot (AC - CD)$.



$$
\begin{aligned}
AC + CD &= AD, \\
\text{and } AC - CD = CB - CD &= DB; \\
\therefore\ (AC + CD) \cdot (AC - CD) &= AD \cdot DB, \\
&= CB^2 - CD^2, \qquad \text{II. 5} \\
&= AC^2 - CD^2
\end{aligned}
$$

ALGEBRAICAL ILLUSTRATION.

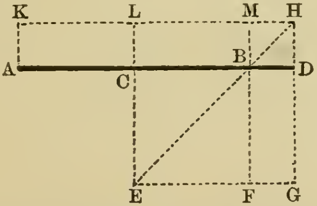
Let $AC = CB = a$, $CD = b$;
then $AD = a + b$, and $DB = a - b$.
Now $AD \cdot DB = (a + b)(a - b) = a^2 - b^2$,
and $CB^2 - CD^2 = a^2 - b^2$;
$\therefore AD \cdot DB = CB^2 - CD^2$.

1. By how much does the rectangle $AC \cdot CB$ exceed the rectangle $AD \cdot DB$? The rectangle contained by the two internal segments of a straight line is the greatest possible when the segments are equal. (Pappus, VII. 13.)
2. The rectangle contained by the two internal segments of a straight line grows less according as the point of section is removed farther from the middle point of the straight line. (Pappus, VII. 14.)
3. Prove that $AC =$ half the sum and $CD =$ half the difference of $AD$ and $DB$.
4. Name two figures in the diagram, each of which = the rectangle contained by half the sum, and half the difference of $AD$ and $DB$.
5. Name that figure in the diagram which is the square on half the sum of $AD$ and $DB$.
6. Name that figure in the diagram which is the square on half the difference of $AD$ and $DB$.

7. Hence show that the proposition may be enunciated: The rectangle contained by any two straight lines is equal to the square on half their sum diminished by the square on half their difference.
8. The perimeter of the rectangle $AD \cdot DB =$ the perimeter of the square on $CB$.
9. Hence show that if a square and a rectangle have equal perimeters, the square has the greater area.
10. Construct a rectangle equal to the difference of two given squares.
11. By means of the first deduction above, and II. 4, show that the sum of the squares on the two segments of a straight line is least when the segments are equal.
12. The square on either of the sides about the right angle of a right-angled triangle, is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

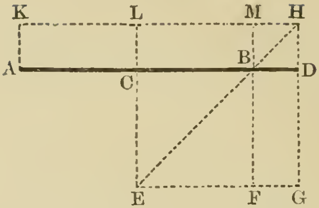
## PROPOSITION 6. THEOREM.

*If a straight line be divided into two equal, and also externally into two unequal segments, the rectangle contained by the unequal segments is equal to the difference between the square on the line between the points of section and the square on half the line.*



Let $AB$ be divided into two equal segments $AC$, $CB$, and also externally into two unequal segments $AD$, $BD$:
*it is required to prove* $AD \cdot DB = CD^2 - CB^2$.

On $CB$ describe the square $CEFB$, and join $BE$. *I.* 46



Through $D$ draw $HDG \parallel CE$, meeting $EB$ and $EF$ produced at $H$ and $G$;
through $H$ draw $HMLK \parallel AB$, meeting $FB$ and $EC$ produced at $M$ and $L$;
and through $A$ draw $AK \parallel CL$. *I.* 31

$$\begin{aligned} \text{Then} \quad AD \cdot DB &= AD \cdot DH, && \textit{II. 7, Cor. 2} \\ &= AH, \\ &= AL + CH, && \textit{I. Ax. 8} \\ &= CM + HF, && \textit{I. 36, 43} \\ &= \text{gnomon } CMG. && \textit{I. Ax. 8} \\ \text{But} \quad CD^2 - CB^2 &= LH^2 - CB^2, && \textit{I. 34} \\ &= LEGH - CEFB, \\ &= \text{gnomon } CMG. && \textit{I. Ax. 8} \end{aligned}$$

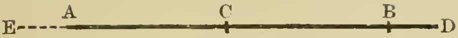
$\therefore AD \cdot DB = CD^2 - CB^2.$

Cor.—The difference of the squares on two straight lines is equal to the rectangle contained by the sum and the difference of the two straight lines.

Let $AC$ and $CD$ be the two straight lines:
*it is required to prove*
$CD^2 - AC^2 = (CD + AC) \cdot (CD - AC)$.

$$\begin{aligned} CD + AC &= AD, \\ \text{and } CD - AC = CD - CB &= DB; \\ \therefore (CD + AC) \cdot (CD - AC) &= AD \cdot DB. \\ &= CD^2 - CB^2, && \textit{II. 6} \\ &= CD^2 - AC^2. \end{aligned}$$

### OTHERWISE: *



Let $AB$ be divided into two equal segments $AC$, $CB$, and also externally into two unequal segments $AD$, $DB$:
*it is required to prove* $AD \cdot DB = CD^2 - CB^2$.

Produce $BA$ to $E$, making $AE = BD$. *I.* 3
Then $EC = CD$, and $EB = AD$.
Now, because $ED$ is divided into two equal segments $EC$, $CD$, and also internally into two unequal segments $EB$, $BD$,
$\therefore\ EB \cdot BD = CD^2 - CB^2$; *II.* 5
$\therefore\ AD \cdot BD = CD^2 - CB^2$.

### ALGEBRAICAL ILLUSTRATION.

Let $AC = CB = a$, $CD = b$;
then $AD = b + a$, and $DB = b - a$.
Now $AD \cdot DB = (b + a)(b - a) = b^2 - a^2$,
and $CD^2 - CB^2 = b^2 - a^2$;
$\therefore\ AD \cdot DB = CD^2 - CB^2$.

1. Does the rectangle $AD \cdot DB$ exceed the rectangle $AC \cdot CB$? Examine the various cases.
2. The rectangle contained by the two external segments of a straight line grows greater according as the point of section is removed farther from the middle point of the straight line.
3. Prove that $AC$ = half the difference, and $CD$ = half the sum of $AD$ and $DB$.
4. Name two figures in the diagram each of which = the rectangle contained by half the sum and half the difference of $AD$ and $DB$.
5. Name that figure in the diagram which is the square on half the sum of $AD$ and $DB$.
6. Name that figure in the diagram which is the square on half the difference of $AD$ and $DB$.

---

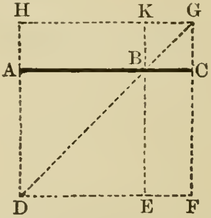
* Due to Mauricius Brescius (of Grenoble), a professor of Mathematics in Paris (probably about the end of the sixteenth century).

7. Hence, show that the proposition may be enunciated: The rectangle contained by any two straight lines is equal to the square on half their sum diminished by the square on half their difference.
8. The perimeter of the rectangle $AD \cdot DB$ = the perimeter of the square on $CD$.

---

## PROPOSITION 7. THEOREM.

*If a straight line be divided externally into any two segments, the square on the straight line is equal to the squares on the two segments diminished by twice the rectangle contained by the segments.*



Let $AB$ be divided externally into any two segments $AC$, $CB$:

*it is required to prove* $AB^2 = AC^2 + CB^2 - 2\,AC \cdot CB$.

On $AB$ describe the square $ADEB$, and join $BD$. *I.* 46
Through $C$ draw $CF \parallel AD$, meeting $DB$ produced at $G$; and through $G$ draw $HK \parallel AB$, meeting $DA$ and $EB$ produced at $H$ and $K$. *I.* 31

Because $CG \parallel AD$, $\therefore \angle CGB = \angle ADB$; *I.* 29
and because $AD = AB$, $\therefore \angle ADB = \angle ABD$; *I.* 5
$\therefore \angle CGB = \angle ABD$,
$= \angle CBG$; *I.* 15
$\therefore CB = CG$. *I.* 6
Hence the $\|^{m}$ $CK$, having two adjacent sides equal, has all its sides equal. *I.* 34

But the $\|^{\text{m}}$ $CK$ has one of its angles, $KBC$, right,
since $\angle KBC = \angle ABE$; *I.* 15
$\therefore$ it has all its angles right; *I.* 34
$\therefore$ the $\|^{\text{m}}$ $CK$ is a square, and $= CB^2$. *I. Def.* 32
Similarly, the $\|^{\text{m}}$ $HF$ is a square, and $= HG^2 = AC^2$.

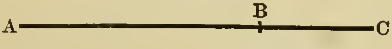
Again, the $\|^{\text{m}}$ $AG = AC \cdot CG = AC \cdot CB$;
$\therefore \quad GE = AC \cdot CB$; *I.* 43
$\therefore \quad AG + GE = 2\,AC \cdot CB$.
Now $\quad AB^2 = ADEB$,
$= HF + CK - AG - GE$, *I. Ax.* 8
$= AC^2 + CB^2 - 2\,AC \cdot CB$.

Cor. 1.—The square on the difference of two straight lines is equal to the sum of the squares on the two straight lines diminished by twice the rectangle contained by the two straight lines.

For if $AC$ and $CB$ be the two straight lines,
then their difference $= AC - CB = AB$.
Now since $\quad AB^2 = AC^2 + CB^2 - 2\,AC \cdot CB$, *II.* 7
$\therefore \quad (AC - CB)^2 = AC^2 + CB^2 - 2\,AC \cdot CB$.

Cor. 2.—The $\|^{\text{ms}}$ about a square's diagonal produced are themselves squares.

### OTHERWISE:



$AB^2 = AB \cdot AC - AB \cdot BC$, *II.* 3
$= (AC \cdot AC - BC \cdot AC) - (AC \cdot BC - BC \cdot BC)$, *II.* 2, 3
$= AC^2 + BC^2 - 2\,AC \cdot BC$.

### ALGEBRAICAL ILLUSTRATION.

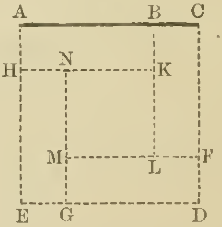
Let $AC = a$, $CB = b$;
then $AB = a - b$.
Now $AB^2 = (a - b)^2 = a^2 - 2ab + b^2$,
and $AC^2 + CB^2 - 2\,AC \cdot CB = a^2 + b^2 - 2ab$;
$\therefore AB^2 = AC^2 + CB^2 - 2\,AC \cdot CB$.

1. Name the two figures which form the sum of the squares on $AC$ and $CB$.
2. Name the figure which is the square on the difference of $AC$ and $CB$.
3. Name the figure which is the difference of the squares on $AB$ and $AC$.
4. Name the figure which is the square on the difference of $AB$ and $AC$.
5. By how much is the square on the difference of $AC$ and $CB$ exceeded by the sum of the squares on $AC$ and $CB$?
6. Show that the proposition may be enunciated: The square on the difference of two straight lines is less than the sum of the squares on the two straight lines by twice the rectangle contained by the two straight lines.
7. The sum of the squares on two straight lines is never less than twice the rectangle contained by the two straight lines.
8. If a straight line be divided internally into two segments, and if twice the rectangle contained by the segments be equal to the sum of the squares on the segments, the straight line is bisected.

---

## PROPOSITION 8. THEOREM.

*The square on the sum of two straight lines diminished by the square on their difference, is equal to four times the rectangle contained by the two straight lines.*



Let $AB$ and $BC$ be two straight lines:

*It is required to prove* $(AB + BC)^2 - (AB - BC)^2$ $4\,AB \cdot BC$.

Place $AB$ and $BC$ in the same straight line,
and on $AC$ describe the square $ACDE$. *I.* 46
From $CD$, $DE$, $EA$ cut off $CF$, $DG$, $EH$ each $= AB$; *I.* 3
through $B$ and $G$ draw $BL$, $GN \parallel AE$,
and through $F$ and $H$ draw $FM$, $HK \parallel AC$. *I.* 31

Then all the $\|^{\text{ms}}$ in the figure are rectangles. *I.* 34, *Cor.*
Now because $CD$, $DE$, $EA$ are each $= AC$, *I. Def.* 32
and $CF$, $DG$, $EH$ are each $= AB$; *Const.*
$\therefore$ $DF$, $EG$, $AH$ are each $= BC$;
$\therefore$ the four rectangles $AK$, $CL$, $DM$, $EN$ are each $= AB \cdot BC$.

Because $AC = AB + BC$,
$\therefore ACDE = AC^2 = (AB + BC)^2$.

Because $BL$, $FM$, $GN$, $HK$ are each $= AB$, *I.* 34
and $BK$, $FL$, $GM$, $HN$ are each $= BC$; *I.* 34
$\therefore$ $KL$, $LM$, $MN$, $NK$ are each $= AB - BC$:
$\therefore$ the rectangle $KLMN$ is a square, and $= (AB - BC)^2$.

$$\text{Hence } (AB + BC)^2 - (AB - BC)^2 = ACDE - KLMN,$$
$$= AK + CL + DM + EN,$$
$$= 4\,AB \cdot BC.$$

### OTHERWISE:

$(AB + BC)^2 = AB^2 + BC^2 + 2\,AB \cdot BC$, *II.* 4, *Cor.* 1
$(AB - BC)^2 = AB^2 + BC^2 - 2\,AB \cdot BC$. *II.* 7, *Cor.* 1

Subtract the second equality from the first;
then $(AB + BC)^2 - (AB - BC)^2 = 4\,AB \cdot BC$.

### ALGEBRAICAL ILLUSTRATION.

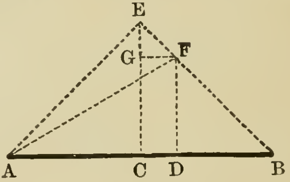
Let $AB = a$, $BC = b$;
then $AB + BC = a + b$, and $AB - BC = a - b$.
Now $(AB + BC)^2 - (AB - BC)^2 = (a + b)^2 - (a - b)^2 = 4ab$,
and $4\,AB \cdot BC = 4ab$;
$\therefore$ $(AB + BC)^2 - (AB - BC)^2 = 4\,AB \cdot BC$.

1. Name the figure which is the square on the sum of $AB$ and $BC$.
2. Name the figure which is the square on the difference of $AB$ and $BC$.
3. Name the figures by which the square on the sum of $AB$ and $BC$ exceeds the square on the difference of $AB$ and $BC$.
4. By how much does the square on the sum of $AB$ and $BC$ exceed the sum of the squares on $AB$ and $BC$?
5. By how much does the sum of the squares on $AB$ and $BC$ exceed the square on the difference of $AB$ and $BC$?

---

## PROPOSITION 9. THEOREM.

*If a straight line be divided into two equal, and also internally into two unequal segments, the sum of the squares on the two unequal segments is double the sum of the squares on half the line and on the line between the points of section.*



Let $AB$ be divided into two equal segments $AC$, $CB$, and also internally into two unequal segments $AD$, $DB$: *it is required to prove* $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

From $C$ draw $CE \perp AB$, and $= AC$ or $CB$, *I.* 11, 3
and join $AE$, $EB$.
Through $D$ draw $DF \parallel CE$, meeting $EB$ at $F$; *I.* 31
through $F$ draw $FG \parallel AB$, meeting $EC$ at $G$; *I.* 31
and join $AF$.

(1) To prove $\angle\, AEB$ right.
Because $\angle\, ACE$ is right,
$\therefore \angle\, CAE + \angle\, CEA =$ a right angle. *I.* 32

But $\angle\ CAE = \angle\ CEA$; *I.* 5
$\therefore$ each of them is half a right angle.
Similarly, $\angle\ CBE$ and $\angle\ CEB$ are each half a right angle;
$\therefore\ \angle\ AEB$ is right.

(2) To prove $EG = GF$.

$\angle\ EGF$ is right, because it $= \angle\ ECB$; *I.* 29
and $\angle\ GEF$ was proved to be half a right angle;
$\therefore\ \angle\ GFE$ is half a right angle; *I.* 32
$\therefore\ \angle\ GEF = \angle\ GFE$;
$\therefore\quad EG = GF$. *I.* 6

(3) To prove $DF = DB$.

$\angle\ FDB$ is right, because it $= \angle\ ECB$; *I.* 29
and $\angle\ DBF$ is half a right angle, being the same as $\angle\ CBE$;
$\therefore\ \angle\ DFB$ is half a right angle; *I.* 32
$\therefore\ \angle\ DBF = \angle\ DFB$;
$\therefore\quad DF = DB$. *I.* 6

$$\begin{aligned}
\text{Now } AD^2 + DB^2 &= AD^2 + DF^2, && (3)\\
&= AF^2, && I.\ 47\\
&= AE^2 + EF^2, && I.\ 47,\ (1)\\
&= AC^2 + CE^2 + EG^2 + GF^2, && I.\ 47\\
&= 2\,AC^2 + 2\,GF^2, && \textit{Const.},\ (2)\\
&= 2\,AC^2 + 2\,CD^2. && I.\ 34
\end{aligned}$$

OTHERWISE:

Consider $AC$ and $CD$ as two straight lines;
then $AD = AC + CD$,
and $DB = CB - CD = AC - CD$.
Hence $AD^2 = (AC + CD)^2 = AC^2 + CD^2 + 2\,AC \cdot CD$, *II.* 4, *Cor.* 1
and $DB^2 = (AC - CD)^2 = AC^2 + CD^2 - 2\,AC \cdot CD$. *II.* 7, *Cor.* 1
Add the second equality to the first;
then $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

ALGEBRAICAL ILLUSTRATION.

Let $AC = CB = a$, $CD = b$;
then $AD = a + b$, and $DB = a - b$.
Now $AD^2 + DB^2 = (a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$,

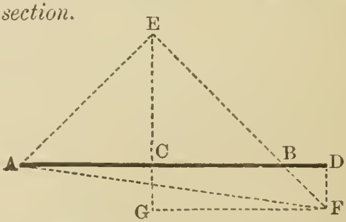
and $2\,AC^2 + 2\,CD^2 = 2a^2 + 2b^2$;
$\therefore AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

1. Show that the proposition may be enunciated: The square on the sum together with the square on the difference of two straight lines = twice the sum of the squares on the two straight lines. Or, The sum of the squares on two straight lines = twice the square on half their sum together with twice the square on half their difference.
2. By how much does $AD^2 + DB^2$ exceed $AC^2 + CB^2$?
3. The sum of the squares on two internal segments of a straight line is the least possible when the straight line is bisected.
4. The sum of the squares on two internal segments of a straight line becomes greater and greater the nearer the point of section approaches either end of the line. (Euclid, x. Lemma before Prop. 43.)
5. Prove that $AD^2 + DB^2 = 4\,CD^2 + 2\,AD \cdot DB$.
6. In the hypotenuse of an isosceles right-angled triangle any point is taken and joined to the opposite vertex; prove that twice the square on this straight line is equal to the sum of the squares on the segments of the hypotenuse.

---

## PROPOSITION 10. THEOREM.

*If a straight line be divided into two equal, and also externally into two unequal segments, the sum of the squares on the two unequal segments is double the sum of the squares on half the line and on the line between the points of section.*



Let $AB$ be divided into two equal segments $AC$, $CB$, and also externally into two unequal segments $AD$, $DB$:

*it is required to prove* $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

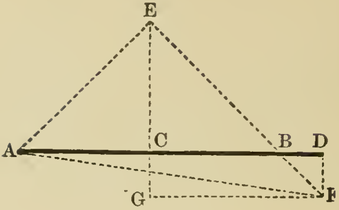
From $C$ draw $CE \perp AB$, and $= AC$ or $CB$, *I.* 11, 3
and join $AE$, $EB$.
Through $D$ draw $DF \parallel CE$, meeting $EB$ produced
at $F$; *I.* 31
through $F$ draw $FG \parallel AB$, meeting $EC$ produced
at $G$; *I.* 31
and join $AF$.

(1) To prove $\angle\ AEB$ right.
Because $\angle\ ACE$ is right,
$\therefore\ \angle\ CAE + \angle\ CEA =$ a right angle. *I.* 32
But $\angle\ CAE = \angle\ CEA$; *I.* 5
$\therefore$ each of them is half a right angle.
Similarly, $\angle\ CBE$ and $\angle\ CEB$ are each half a right angle;
$\therefore\ \angle\ AEB$ is right.

(2) To prove $EG = GF$.
$\angle\ EGF$ is right, because it $= \angle\ ECB$; *I.* 29
and $\angle\ GEF$ was proved to be half a right angle;
$\therefore\ \angle\ GFE$ is half a right angle; *I.* 32
$\therefore\ \angle\ GEF = \angle\ GFE$;
$\therefore\quad EG = GF$. *I.* 6

(3) To prove $DF = DB$.
$\angle\ FDB$ is right, because it $= \angle\ ECB$; *I.* 29
and $\angle\ DBF$ is half a right angle, being $= \angle\ CBE$; *I.* 15
$\therefore\ \angle\ DFB$ is half a right angle; *I.* 32
$\therefore\ \angle\ DBF = \angle\ DFB$;
$\therefore\quad DF = DB$. *I.* 6

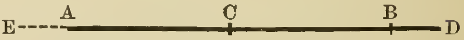
Now $AD^2 + DB^2 = AD^2 + DF^2$, (3)
$= AF^2$, *I.* 47
$= AE^2 + EF^2$, *I.* 47, (1)
$= AC^2 + CE^2 + EG^2 + GF^2$, *I.* 47
$= 2\,AC^2 + 2\,GF^2$, *Const.*, (2)
$= 2\,AC^2 + 2\,CD^2$. *I.* 34



OTHERWISE:

Consider $AC$ and $CD$ as two straight lines;
then $AD = CD + AC$,
and $DB = CD - CB = CD - AC$.
Hence $AD^2 = (CD + AC)^2 = CD^2 + AC^2 + 2\,CD \cdot AC$; *II.* 4, *Cor.* 1
and $DB^2 = (CD - AC)^2 = CD^2 + AC^2 - 2\,CD \cdot AC$. *II.* 7, *Cor.* 1
Add the second equality to the first;
then $AD^2 + DB^2 = 2\,CD^2 + 2\,AC^2$.

OR: *



Let $AB$ be divided into two equal segments $AC$, $CB$, and also externally into two unequal segments $AD$, $DB$:
*it is required to prove* $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

Produce $BA$ to $E$, making $AE = BD$. *I.* 3
Then $EC = CD$, and $EB = AD$.
Now because $ED$ is divided into two equal segments $EC$, $CD$, and also internally into two unequal segments $EB$, $BD$;
$\therefore$ $EB^2 + BD^2 = 2\,EC^2 + 2\,CB^2$; *II.* 9
$\therefore$ $AD^2 + BD^2 = 2\,CD^2 + 2\,AC^2$.

ALGEBRAICAL ILLUSTRATION.

Let $AC = CB = a$, $CD = b$;
then $AD = b + a$, and $DB = b - a$.
Now $AD^2 + DB^2 = (b + a)^2 + (b - a)^2 = 2b^2 + 2a^2$,
and $2\,AC^2 + 2\,CD^2 = 2\,a^2 + 2b^2$;
$\therefore$ $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

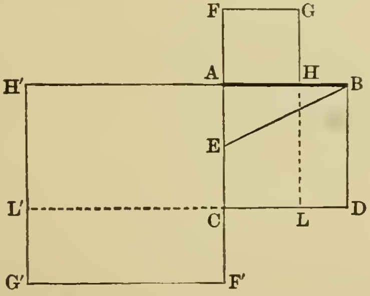
---

* *Clavii Commentaria in Euclidis Elementa Geometrica* (1612), p. 93.

1. Show that the proposition may be enunciated: The square on the sum together with the square on the difference of two straight lines = twice the sum of the squares on the two straight lines. Or, The sum of the squares on two straight lines = twice the square on half their sum together with twice the square on half their difference.
2. By how much does $AD^2 + DB^2$ exceed $AC^2 + CB^2$?
3. The sum of the squares on two external segments of a straight line becomes less and less the nearer the point of section approaches either end of the line.
4. Prove that $AD^2 + DB^2 = 4\,CD^2 - 2\,AD \cdot DB$.
5. In the hypotenuse produced of an isosceles right-angled triangle, any point is taken and joined to the opposite vertex; prove that twice the square on this straight line is equal to the sum of the squares on the segments of the hypotenuse.

## PROPOSITION 11. Problem.

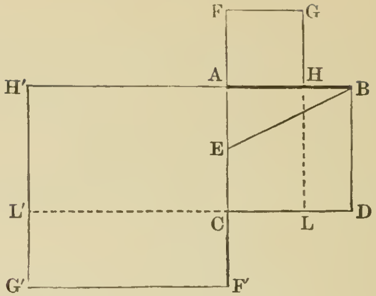
*To divide a given straight line internally and externally* in medial section.*



Let $AB$ be the given straight line:
*it is required to divide it in medial section.*

---

* The second part of this proposition is not given by Euclid.



(1) Internally:

On $AB$ describe the square $ABDC$. *I.* 46
Bisect $AC$ at $E$; *I.* 10
join $EB$, and produce $CA$ to $F$, making $EF = EB$. *I.* 3
On $AF$ (the difference of $EF$ and $EA$) describe the square $AFGH$. *I.* 46

$H$ is the point required.

Complete the rectangle $FL$.

Because $CA$ is divided into two equal segments $CE$, $EA$, and also externally into two unequal segments $CF$, $FA$;

∴ $CF \cdot FA = EF^2 - EA^2$, *II.* 6
$= EB^2 - EA^2$,
$= AB^2$; *I.* 47, *Cor.*
that is, $CF \cdot FG = AB^2$;
that is, $CG = AD$.
From each of these equals take $AL$;
∴ $FH = HD$;
that is, $AH^2 = DB \cdot BH$,
$= AB \cdot BH$.

(2) Externally:

On $AB$ describe the square $ABDC$. *I.* 46
Bisect $AC$ at $E$; *I.* 10
join $EB$, and produce $AC$ to $F'$, making $EF' = EB$. *I.* 3

On $AF'$ (the sum of $EF'$ and $EA$) describe the square $AF'G'H'$, I. 46

$H'$ is the point required.

Complete the rectangle $F'L'$.

Because $CA$ is divided into two equal segments $CE$, $EA$, and also externally into two unequal segments $CF'$, $F'A$;

$$\begin{aligned} \therefore \quad CF' \cdot F'A &= EF'^2 - EA^2, && \text{II. 6} \\ &= EB^2 - EA^2, \\ &= AB^2; && \text{I. 47, Cor.} \end{aligned}$$

that is, $CF' \cdot F'G' = AB^2$;

that is, $CG' = AD$.

To each of these equals add $AL'$;

$$\therefore \quad F'H' = H'D;$$

$$\begin{aligned} \text{that is,} \quad AH'^2 &= DB \cdot BH', \\ &= AB \cdot BH'. \end{aligned}$$

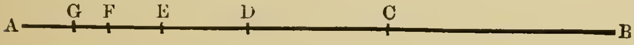
Cor. 1.—If a straight line be divided internally in medial section, and from the greater segment a part be cut off equal to the less segment, the greater segment will be divided in medial section.

For in the proof of the proposition it has been shown that $CF \cdot FA = AB^2$, that is $= AC^2$;

$\therefore$ $CF$ is divided internally in medial section at $A$.

Now, from $AB$, which $= AC$, the greater segment of $CF$, a part $AH$ has been cut off $= AF$, the less segment of $CF$; and $AB$ has been shown to be divided in medial section at $H$.

Let $AB$ be divided internally in medial section at $C$, so that $AC$ is the greater segment.



From $AC$ cut off $AD = BC$; then $AC$ is divided in medial section at $D$, and $AD$ is the greater segment.

From $AD$ cut off $AE = CD$; then $AD$ is divided in medial section at $E$, and $AE$ is the greater segment.

From $AE$ cut off $AF = DE$; then $AE$ is divided in medial section at $F$, and $AF$ is the greater segment.

From $AF$ cut off $AG = EF$; then $AF$ is divided in medial section at $G$, and $AG$ is the greater segment.

This process may evidently be continued as long as we please, and it will be seen on comparison that it is equivalent to the arithmetical method of finding the greatest common measure. That method, if applied to two integers, always, however, comes to an end; unity, in default of any other number, being always a common measure of any two integers. In like manner any two fractions, whether vulgar or decimal, have always some common measure, for instance, unity divided by their least common denominator. From these considerations, therefore, it will appear that the segments of a straight line divided in medial section cannot both be expressed exactly either in integers or fractions; in other words, these segments are incommensurable.

Cor. 2.—If a straight line be divided internally in medial section, and to the given straight line a part be added equal to the greater segment, the whole straight line will be divided in medial section.

For this process is just the reversal of that described in Cor. 1, as will be evident from the following. (See fig. to Cor. 1.)

Let $AF$ be divided in medial section at $G$, so that $AG$ is the greater segment.

To $AF$ add $FE$, which $= AG$; then $AE$ is divided in medial section at $F$, and $AF$ is the greater segment.

To $AE$ add $ED$, which $= AF$; then $AD$ is divided in medial section at $E$, and $AE$ is the greater segment.

To $AD$ add $DC$, which $= AE$; then $AC$ is divided in medial section at $D$, and $AD$ is the greater segment.

To $AC$ add $CB$, which $= AD$; then $AB$ is divided in medial section at $C$, and $AC$ is the greater segment.

### ALGEBRAICAL APPLICATION.

Let $AB = a$; to find the length of $AH$ or $AH'$.

Denote $AH$ by $x$; then $BH = a - x$.

Now, since $AB \cdot BH = AH^2$

$\therefore a(a - x) = x^2$, a quadratic equation, which being solved gives

$$x = \frac{a(\sqrt{5} - 1)}{2} \text{ or } \frac{-a(\sqrt{5} + 1)}{2}.$$

The first value of $x$, which is less than $a$, since $\frac{\sqrt{5}-1}{2}$ is less than unity, corresponds to $AH$; and the second value of $x$, which is numerically greater than $a$, since $\frac{\sqrt{5}+1}{2}$ is greater than unity, corresponds to $AH'$. The significance of the $-$ in the second value cannot be explained here; it will be enough to say that it indicates that $AH$ and $AH'$ are measured in opposite directions from $A$.

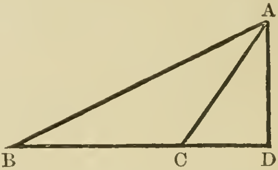
The following approximation to the values of the segments of a straight line divided internally in medial section, is given in Leslie's *Elements of Geometry* (4th edition, p. 312), and attributed to Girard, a Flemish mathematician (17th cent.).

Take the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c., where each term is got by taking the sum of the preceding two. If any term be considered as denoting the length of the straight line, the two preceding terms will approximately denote the lengths of its segments when it is divided internally in medial section. Thus, if 89 be the length of the line, its segments will be nearly 34 and 55; because $89 \times 34 = 3026$, and $55^2 = 3025$. If 144 be the length of the line, its segments will be nearly 55 and 89; because $144 \times 55 = 7920$, and $89^2 = 7921$.

1. It is assumed in the construction that a side of the square described on $AF$ will coincide with $AB$. Prove this.
2. If $AB \cdot BH = AH^2$, prove that $AH$ is greater than $BH$.
3. If $CH$ be produced, it will cut $BF$ at right angles.
4. The point of intersection of $BE$ and $CH$ is the projection of $A$ on $CH$.
5. It is assumed in the proof of the second part that a side of the square described on $AF'$ will be in the same straight line with $AB$. Prove this.
6. If $AB \cdot BH' = AH'^2$, prove that $AH'$ is greater than $AB$.
7. If $CH'$ be produced, it will cut $BF'$ at right angles.
8. The point of intersection of $BE$ and $CH'$ is the projection of $A$ on $CH'$.
9. Prove that $HB$ is divided externally in medial section at $A$, and $H'B$ internally at $A$.
10. Hence name all the straight lines in the figure that are divided internally or externally in medial section.

## PROPOSITION 12. THEOREM.

*In obtuse-angled triangles, the square on the side opposite the obtuse angle is equal to the sum of the squares on the other two sides increased by twice the rectangle contained by either of those sides and the projection on it of the other side.*



Let $ABC$ be an obtuse-angled triangle, having the obtuse angle $ACB$; and let $CD$ be the projection of $CA$ on $BC$: *it is required to prove* $AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD$.

Because $BD$ is divided internally into any two segments $BC$, $CD$,

$\therefore\ BD^2 = BC^2 + CD^2 + 2\,BC \cdot CD.$ *II.* 4

Adding $DA^2$ to both sides,

$$BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2\,BC \cdot CD;$$

$\therefore\quad AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD.$ *I.* 47

### ALGEBRAICAL APPLICATION.

Let the sides opposite the $\angle$s $A$, $B$, $C$ be denoted by $a$, $b$, $c$, so that $AB = c$, $BC = a$, $CA = b$;

then, since $AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD$, *II.* 12

$$c^2 = a^2 + b^2 + 2a \cdot CD;$$

$$\therefore \qquad CD = \frac{c^2 - a^2 - b^2}{2a};$$

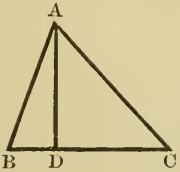
$$\therefore \qquad BD = BC + CD = a + \frac{c^2 - a^2 - b^2}{2a} = \frac{a^2 - b^2 + c^2}{2a}.$$

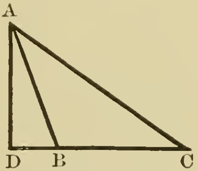
Hence, if the three sides of an obtuse-angled triangle are known, we can calculate the lengths of the segments into which either side about the obtuse angle is divided by a perpendicular from one of the acute angles.

1. If from $B$ there be drawn $BE \perp AC$ produced, then $BC \cdot CD = AC \cdot CE$.
2. $ABCD$ is a $\parallel^{m}$ having $\angle ABC$ equal to an angle of an equilateral triangle; prove $BD^2 = BC^2 + CD^2 + BC \cdot CD$.
3. If $AB^2 = AC^2 + 3\,CD^2$ (figure to proposition), how will the perpendicular $AD$ divide $BC$?
4. If $\angle ACB$ become more and more obtuse, till at length $A$ falls on $BC$ produced, what does the proposition become?

## PROPOSITION 13. THEOREM.

*In every triangle the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by either of those sides and the projection on it of the other side.*





Let $ABC$ be any triangle, having the acute angle $ACB$; and let $CD$ be the projection of $CA$ on $BC$:
*it is required to prove* $AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD$.

Because $BD$ is divided externally into any two segments $BC$, $CD$,

$$\therefore \quad BD^2 = BC^2 + CD^2 - 2\,BC \cdot CD. \qquad \textit{II. 7}$$

Adding $DA^2$ to both sides,

$$BD^2 + DA^2 = BC^2 + CD^2 + DA^2 - 2\,BC \cdot CD;$$

$$\therefore \quad AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD. \qquad \textit{I. 47}$$

ALGEBRAICAL APPLICATION.

As before, let $AB = c$, $BC = a$, $CA = b$;
then, since $AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD$, *II.* 13

$$c^2 = a^2 + b^2 - 2a \cdot CD;$$

$$\therefore \qquad CD = \frac{a^2 + b^2 - c^2}{2a};$$

$$\therefore \text{(fig. 1)} \qquad BD = BC - CD = a - \frac{a^2 + b^2 - c^2}{2a} = \frac{a^2 - b^2 + c^2}{2a};$$

$$\text{and (fig. 2)} \qquad BD = CD - BC = \frac{a^2 + b^2 - c^2}{2a} - a = \frac{b^2 - c^2 - a^2}{2a}.$$

Hence, from the results of this proposition and the preceding, if the three sides of any triangle are known, we can calculate the lengths of the segments into which any side is divided by a perpendicular from the opposite angle.

Hence, again, if the three sides of any triangle are known, we can calculate the length of the perpendicular drawn from any angle of a triangle to the opposite side.

For example (fig. 1), to find the length of $AD$.

$$AD^2 = AC^2 - CD^2, \qquad \textit{I. 47, Cor.}$$

$$= b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2$$

$$= \frac{4a^2 b^2 - (a^2 + b^2 - c^2)^2}{4a^2},$$

$$= \frac{(2ab + a^2 + b^2 - c^2)\,(2ab - a^2 - b^2 + c^2)}{4a^2},$$

$$= \frac{\{(a^2 + 2ab + b^2) - c^2\}\,\{c^2 - (a^2 - 2ab + b^2)\}}{4a^2}$$

$$= \frac{\{(a + b)^2 - c^2\}\,\{c^2 - (a - b)^2\}}{4a^2},$$

$$= \frac{(a + b + c)\,(a + b - c)\,(c + a - b)\,(c - a + b)}{4a^2};$$

$$\therefore AD = \frac{1}{2a}\sqrt{(a + b + c)\,(a + b - c)\,(a - b + c)\,(b + c - a)}.$$

This expression for the length of $AD$ may be put into a shorter and more convenient form, thus:

Denote the semi-perimeter of the $\triangle\ ABC$ by $s$;

then $a + b + c =$ the perimeter $= 2s$;

$\therefore a + b - c = a + b + c - 2c = 2s - 2c = 2\,(s - c)$,

$a - b + c = a + b + c - 2b = 2s - 2b = 2\,(s - b)$,

and $b + c - a = a + b + c - 2a = 2s - 2a = 2\,(s - a)$.

Hence $AD = \frac{1}{2a}\sqrt{2s \cdot 2(s-c) \cdot 2(s-b) \cdot 2(s-a)}$,

$$= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, the perpendicular from $B$ on $CA = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$

and " " $C$ on $AB = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

Hence, lastly, if the three sides of a triangle are known, we can calculate the area of the triangle.

For the area of $\triangle\ ABC = \frac{1}{2} BC \cdot AD$, *I.* 41, 35

$$= \frac{a}{2} \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)},$$

$$= \sqrt{s(s-a)(s-b)(s-c)};$$

which expression may be put into the form of a rule, thus :

From half the sum of the three sides, subtract each side separately ; multiply the half sum and the three remainders together, and the square root of the product will be the area.*

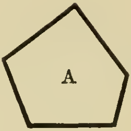
1. If from $B$ there be drawn $BE \perp AC$ or $AC$ produced, then $BC \cdot CD = AC \cdot CE$.
2. $ABCD$ is a ||m having $\angle\ ABC$ double of an angle of an equilateral triangle ; prove $BD^2 = BC^2 + CD^2 - BC \cdot CD$.
3. If $AB^2 = AC^2 + 3\,CD^2$ (fig. 1 to proposition), how will the perpendicular $AD$ divide $BC$?
4. If $\angle\ ACB$ become more and more acute till at length $A$ falls on $CB$ or $CB$ produced, what does the proposition become ?
5. If the square on one side of a triangle be greater than the sum of the squares on the other two sides, the angle contained by these two sides is obtuse. (Converse of II. 12.)
6. If the square on one side of a triangle be less than the sum of the squares on the other two sides, the angle contained by these two sides is acute. (Converse of II. 13.)
7. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either of the equal sides and the projection on it of the base.

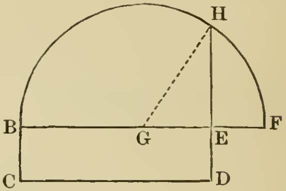
---

* The discovery of this expression for the area of a triangle is due to Heron of Alexandria. See Hultsch's *Heronis Alexandrini . . . reliquiæ* (Berlin, 1864), pp. 235–237.

## PROPOSITION 14. PROBLEM.

*To describe a square that shall be equal to a given rectilineal figure.*





Let $A$ be the given rectilineal figure:
*it is required to describe a square* $= A$.

Describe the rectangle $BCDE = A$. *I.* 45
Then, if $BE = ED$, the rectangle is a square, and what was required is done.
But if not, produce $BE$ to $F$, making $EF = ED$. *I.* 3
Bisect $BF$ in $G$; *I.* 10
with centre $G$ and radius $GF$ describe the semicircle $BHF$;
and produce $DE$ to $H$. $EH^2 = A$.

Join $GH$.

Because $BF$ is divided into two equal segments $BG$, $GF$, and also internally into two unequal segments $BE$, $EF$;

$$\therefore \quad BE \cdot EF = GF^2 - GE^2, \qquad II.\ 5$$
$$= GH^2 - GE^2,$$
$$= EH^2. \qquad I.\ 47,\ Co$$
$$\therefore \quad BD = EH^2;$$
$$.\quad A = EH^2.$$

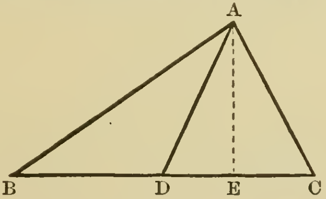
1. From any point in the arc of a semicircle, a perpendicular is drawn to the diameter. Prove that the square on this perpendicular = the rectangle contained by the segments into which it divides the diameter.

2. Divide a given straight line internally into two segments, such that the rectangle contained by them may be equal to the square on another given straight line. What limits are there to the length of the second straight line?
3. Divide a given straight line externally into two segments, such that the rectangle contained by them may be equal to the square on another given straight line. Are there any limits to the length of the second straight line?
4. Describe a rectangle equal to a given square, and having one of its sides equal to a given straight line.

# APPENDIX II.

## PROPOSITION 1.

*The sum of the squares on two sides of a triangle is double the sum of the squares on half the base and on the median to the base.*[*]



Let $ABC$ be a triangle, $AD$ the median to the base $BC$: *it is required to prove* $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$.

Draw $AE \perp BC$. *I.* 12

Then $AB^2 = BD^2 + AD^2 + 2\,BD \cdot DE$, *II.* 12

and $AC^2 = CD^2 + AD^2 - 2\,CD \cdot DE$. *II.* 13

But $BD^2 = CD^2$, and $BD \cdot DE = CD \cdot DE$, since $BD = CD$;

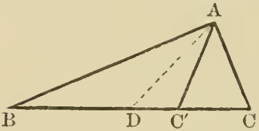
$\therefore\ AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$.

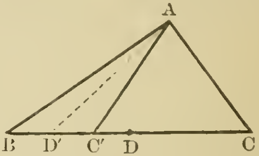
Cor.—The theorem is true, however near the vertex $A$ may be to the base $BC$. When $A$ falls on $BC$, the theorem becomes II. 9; when $A$ falls on $BC$ produced, the theorem becomes II. 10.

---

[*] Pappus, VII. 122.

Note.—It may be well to remark that the converse of the theorem, 'If $ABC$ be a triangle, and from the vertex $A$ a straight line $AD$ be drawn to the base $BC$, so that $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$, then $D$ is the middle point of $BC$,' is not always true.





For, let $ABC$, $ABC'$ be two triangles having $AC = AC'$.

Find $D$, the middle point of $BC$. $D$ must fall either between $B$ and $C'$, between $C$ and $C'$, or on $C'$.

In the first case, join $AD$.

Then $$AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2; \qquad \textit{App. II. } 1$$

$$\therefore \quad AB^2 + AC'^2 = 2\,BD^2 + 2\,AD^2;$$

and we know that $D$ is not the middle point of $BC'$.

In the second case, find $D'$ the middle point of $BC'$, and join $AD'$.

Then $$AB^2 + AC'^2 = 2\,BD'^2 + 2\,AD'^2; \qquad \textit{App. II. } 1$$

$$\therefore \quad AB^2 + AC^2 = 2\,BD'^2 + 2\,AD'^2;$$

and we know that $D'$ is not the middle point of $BC$.

The third case needs no discussion.

## Proposition 2.

*The difference of the squares on two sides of a triangle is double the rectangle contained by the base and the distance of its middle point from the perpendicular on it from the vertex.**

Fig. 1.

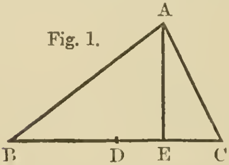
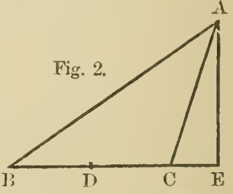


Fig. 2.



Let $ABC$ be a triangle, $D$ the middle point of the base $BC$, and $AE$ the perpendicular from $A$ on $BC$:

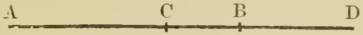
*it is required to prove* $AB^2 - AC^2 = 2\,BC \cdot DE$.

* Pappus, VII. 120.

$$\begin{aligned}
\text{For}\quad AB^2 - AC^2 &= (BE^2 + AE^2) - (EC^2 + AE^2), && \textit{I. } 47\\
&= BE^2 - EC^2,\\
&= (BE + EC)(BE - EC), && \textit{II. } 5, 6, \textit{Corr.}\\
&= BC \cdot 2\,DE \quad \text{in fig. 1};\\
\text{or} &= 2\,DE \cdot BC \quad \text{in fig. 2},\\
&= 2\,BC \cdot DE.
\end{aligned}$$

### Proposition 3.

*If the straight line AD be divided internally at any two points C and B, then $AC \cdot BD + AD \cdot BC = AB \cdot CD$.*[*]

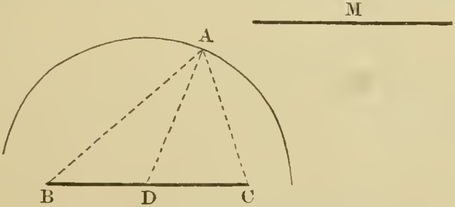


$$\begin{aligned}
\text{For } AC \cdot BD + AD \cdot BC &= AC \cdot BD + (BD + AB) \cdot BC,\\
&= AC \cdot BD + BD \cdot BC + AB \cdot BC, && \textit{II. } 1\\
&= BD \cdot (AC + BC) \quad + AB \cdot BC, && \textit{II. } 1\\
&= BD \cdot AB \quad + AB \cdot BC,\\
&= AB \cdot (BD + BC), && \textit{II. } 1\\
&= AB \cdot CD.
\end{aligned}$$

## LOCI.

### Proposition 4.

*Find the locus of the vertices of all the triangles which have the same base and the sum of the squares of their sides equal to a given square.*



Let $BC$ be the given base, $M^2$ the given square.

Suppose $A$ to be a point situated on the required locus.

Join $AB$, $AC$;

bisect $BC$ in $D$, and join $AD$. *I.* 10

---

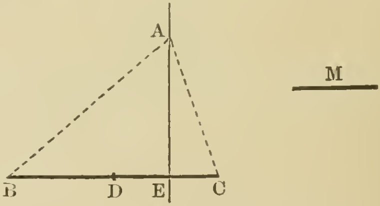
[*] Euler, *Novi Comm. Petrop.*, vol. i. p. 49.

Then, since $A$ is a point on the locus, $AB^2 + AC^2 = M^2$. *Hyp.*
But $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$; *App. II.* 1
$\therefore\ 2\,BD^2 + 2\,AD^2 = M^2$;
$\therefore\ AD^2 = \frac{1}{2}M^2 - BD^2$.
Now $\frac{1}{2}M^2$ is a constant magnitude, and so is $BD^2$, being the square on half the given base;
$\therefore\ \frac{1}{2}M^2 - BD^2$ must be constant;
$\therefore\ AD^2$ must be constant.
And since $AD^2$ is constant, $AD$ must be equal to a fixed length; that is, the vertex of any triangle fulfilling the given conditions is always at a constant distance from a fixed point $D$, the middle of the given base. Hence, the locus required is the ○ᶜᵉ of a circle whose centre is the middle point of the base.

To determine the locus completely, it would be necessary to find the length of the radius of the circle. This may be left to the reader.

## Proposition 5.

*Find the locus of the vertices of all the triangles which have the same base, and the difference of the squares of their sides equal to a given square.*



Let $BC$ be the given base, $M^2$ the given square.
Suppose $A$ to be a point situated on the required locus.
Join $AB$, $AC$;
bisect $BC$ in $D$, and draw $AE \perp BC$ or $BC$ produced. *I.* 10, 12
Then, since $A$ is a point on the locus $AB^2 - AC^2 = M^2$. *Hyp.*
But $AB^2 - AC^2 = 2\,BC \cdot DE$; *App. II.* 2
$\therefore\ 2\,BC \cdot DE = M^2$.
Now $M^2$ is a constant magnitude, and so is $2\,BC$;
$\therefore\ DE$ must be constant;
$\therefore$ a perpendicular drawn to $BC$ from the vertex of any triangle fulfilling the given conditions will cut $BC$ at a fixed point.

If $AC^2 - AB^2 = M^2$, the perpendicular from $A$ on $BC$ will cut $BC$ at a point $E'$ on the other side of $D$, such that $DE' = DE$.

Hence, the locus consists of two straight lines drawn perpendicular to the base and equally distant from the middle point of the base.

### DEDUCTIONS.

1. If from the vertex of an isosceles triangle a straight line be drawn to cut the base either internally or externally, the difference between the squares on this line and either side is equal to the rectangle contained by the segments of the base. (Pappus, III. 5.)
2. The sum of the squares on the diagonals of a $\parallel^{m}$ is equal to the sum of the squares on the four sides.
3. The sum of the squares on the diagonals of any quadrilateral is equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.
4. The sum of the squares on the four sides of any quadrilateral exceeds the sum of the squares on the two diagonals by four times the square on the straight line which joins the middle points of the diagonals. (Euler, *Novi Comm. Petrop.*, i. p. 66.)
5. The centre of a fixed circle is the middle point of the base of a triangle. If the vertex of the triangle be on the $\bigcirc^{ce}$, the sum of the squares on the two sides of the triangle is constant.
6. The centre of a fixed circle is the point of intersection of the diagonals of a $\parallel^{m}$. Prove that the sum of the squares on the straight lines drawn from any point on the $\bigcirc^{ce}$ to the four vertices of the $\parallel^{m}$ is constant.
7. Two circles are concentric. Prove that the sum of the squares of the distances from any point on the $\bigcirc^{ce}$ of one of the circles to the ends of a diameter of the other is constant.
8. The middle point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.
9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the three medians, or equal to nine times the sum of the squares on the straight lines which join the centroid to the three vertices.
10. If $ABCD$ be a quadrilateral, and $P$, $Q$, $R$, $S$ be the middle points of $AB$, $BC$, $CD$, $DA$ respectively, then $2\,PR^2 + AB^2 + CD^2 = 2\,QS^2 + BC^2 + DA^2$.

11. Thrice the sum of the squares on the sides of any pentagon = the sum of the squares on the diagonals together with four times the sum of the squares on the five straight lines joining, in order, the middle points of those diagonals.

12. If $A$, $B$ be fixed points, and $O$ any other point, the sum of the squares on $OA$ and $OB$ is least when $O$ is the middle point of $AB$.

13. Prove II. 9, 10 by the following construction: On $AD$ describe a rectangle $AEFD$ whose sides $AE$, $DF$ are each $= AC$ or $CB$. According as $D$ is in $AB$, or in $AB$ produced, from $DF$, or $DF$ produced, cut off $FG = DB$; and join $EC$, $CG$, $GE$. Show how these figures may be derived from those in the text.

14. If from the vertex of the right angle of a right-angled triangle a perpendicular be drawn to the hypotenuse, then (1) the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse; (2) the square on either side is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side.

15. The sum of the squares on two unequal straight lines is greater than twice the rectangle contained by the straight lines.

16. The sum of the squares on three unequal straight lines is greater than the sum of the rectangles contained by every two of the straight lines.

17. The square on the sum of three unequal straight lines is greater than three times the sum of the rectangles contained by every two of the straight lines.

18. The sum of the squares on the sides of a triangle is less than twice the sum of the rectangles contained by every two of the sides.

19. If one side of a triangle be greater than another, the median drawn to it is less than the median drawn to the other.

20. If a straight line $AB$ be bisected in $C$, and divided internally at $D$ and $E$, $D$ being nearer the middle than $E$, then $AD \cdot DB = AE \cdot EB + CD \cdot DE + CE \cdot ED$.

21. $ABC$ is an isosceles triangle having each of the angles $B$ and $C = 2A$. $BD$ is drawn $\perp AC$; prove $AD^2 + DC^2 = 2BD^2$.

22. Divide a given straight line internally so that the squares on the whole and on one of the segments may be double of the square on the other segment.

23. Given that $AB$ is divided internally at $H$, and externally at $H'$, in medial section, prove the following:

(1) $AH \cdot BH = (AH + BH) \cdot (AH - BH)$;
$AH' \cdot BH' = (BH' + AH') \cdot (BH' - AH')$.
(2) $AH \cdot (AH - BH) = BH^2$; $AH' \cdot (AH' + BH') = BH'^2$.
(3) $AB^2 + BH^2 = 3\,AH^2$; $AB^2 + BH'^2 = 3\,AH'^2$.
(4) $(AB + BH)^2 = 5\,AH^2$; $(AB + BH')^2 = 5\,AH'^2$.
(5) $(AH - BH)^2 = 3\,BH^2 - AH^2$; $(BH' - AH')^2 = 3\,AH'^2 - BH'^2$.
(6) $(AH + BH)^2 = 3\,AH^2 - BH^2$; $(AH' + BH')^2 = 3\,BH'^2 - AH'^2$.
(7) $(AB + AH)^2 = 8\,AH^2 - 3\,BH^2$; $(AH' - AB)^2 = 8\,AH'^2 - 3\,BH'^2$.
(8) $AB^2 + AH^2 = 4\,AH^2 - BH^2$; $AB^2 + AH'^2 = 4\,AH'^2 - BH'^2$.

24. In any triangle $ABC$, if $BP$, $CQ$ be drawn $\perp$ $CA$, $BA$, produced if necessary, then shall $BC^2 = AB \cdot BQ + AC \cdot CP$.

25. If from the hypotenuse of a right-angled triangle segments be cut off equal to the adjacent sides, the square of the middle segment thus formed = twice the rectangle contained by the extreme segments. Show how this theorem may be used to find numbers expressing the sides of a right-angled triangle. (Leslie's *Elements of Geometry*, 1820, p. 315.)

## Loci.

1. Given a $\triangle$ $ABC$; find the locus of the points the sum of the squares of whose distances from $B$ and $C$, the ends of the base, is equal to the sum of the squares of the sides $AB$, $AC$.
2. Given a $\triangle$ $ABC$; find the locus of the points the difference of the squares of whose distances from $B$ and $C$, the ends of the base, is equal to the difference of the squares of the sides $AB$, $AC$.
3. Of the $\triangle$ $ABC$, the base $BC$ is given, and the sum of the sides $AB$, $AC$; find the locus of the point where the perpendicular from $C$ to $AC$ meets the bisector of the exterior vertical angle at $A$.
4. Of the $\triangle$ $ABC$, the base $BC$ is given, and the difference of the sides $AB$, $AC$; find the locus of the point where the perpendicular from $C$ to $AC$ meets the bisector of the interior vertical angle at $A$.
5. A variable chord of a given circle subtends a right angle at a fixed point; find the locus of the middle point of the chord. Examine the cases when the fixed point is inside the circle, outside the circle, and on the $\bigcirc^{ce}$.

# BOOK III.

## DEFINITIONS.

1. A **circle** is a plane figure contained by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal. This point is called the **centre** of the circle, and the straight lines drawn from the centre to the circumference are called **radii.**

Cor. 1.—If a point be situated inside a circle, its distance from the centre is less than a radius; and if it be situated outside, its distance from the centre is greater than a radius

Thus, in fig. 1, *OP*, the distance of the point *P* from the centre *O*, is less than the radius *OA*; in fig. 2, *OP* is greater than the radius *OA*.

Fig. 1.

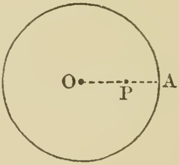
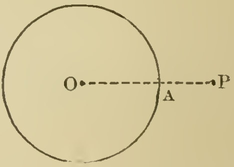


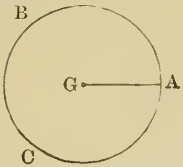
Fig. 2.

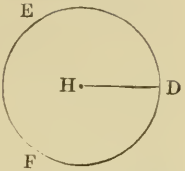


Cor. 2.—Conversely, if the distance of a point from the centre of a circle be less than a radius, the point must be situated inside the circle; if its distance from the centre be greater than a radius, it must be situated outside the circle.

Cor. 3.—If the radii of two circles be equal, the circumferences are equal, and so are the circles themselves.

This may be rendered evident by applying the one circle to the other, so that their centres shall coincide. Since the radii of the one circle are equal to those of the other, every point in the circumference of the one circle will coincide with a point in the circumference of the other; therefore, the two circumferences coincide and are equal. Consequently also the two circles coincide and are equal.





Cor. 4.—Conversely, if two circles be equal, their radii are equal, and also their circumferences.

This may be proved indirectly, by supposing the radii unequal.

Cor. 5.—A circle is given in magnitude when the length of its radius is given, and a circle is given in position and magnitude when the position of its centre and the length of its radius are given. (Euclid's *Data*, Definitions 5 and 6.)

Cor. 6.—The two parts into which a diameter divides a circle are equal.

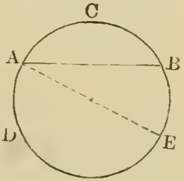
This may be proved, like Cor. 3, by superposition.

The two parts are therefore called *semicircles*.

Cor. 7.—The two parts into which a straight line not a diameter divides a circle are unequal.

Thus if $AB$ is not a diameter of the circle $ABC$, the two parts $ACB$ and $ADB$ into which $AB$ divides the circle are unequal.

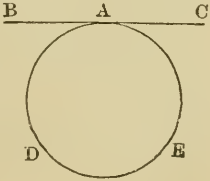
For if a diameter $AE$ be drawn, the part $ACB$ is less than the semicircle $ABE$, and the part $ADB$ is greater than the semicircle $ADE$.



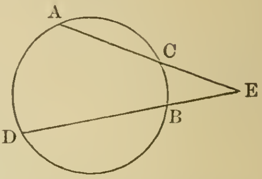
2. **Concentric** circles are those which have a common centre.

3. A straight line is said to **touch** a circle, or to be a **tangent** to it, when it meets the circle, but being produced does not cut it.

Thus $BC$ is a tangent to the circle $ADE$.



4. A straight line drawn from a point outside a circle, and cutting the circumference, is called a **secant.**



Thus $ECA$ and $EBD$ are secants of the circle $ABC$.

If the secant $ECA$ were, like one of the hands of a watch, to revolve round $E$ as a pivot, the points $A$ and $C$ would approach one another, and at length coincide. When the points $A$ and $C$ coincided, the secant would have become a tangent. Hence a tangent to a circle may be defined to be a secant in its limiting position, or a secant which meets the circle in two coincident points.

This way of regarding a tangent straight line may be applied also to a tangent circle.

5. Circles which meet but do not cut one another, are said to **touch** one another.

Fig. 1.

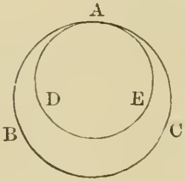
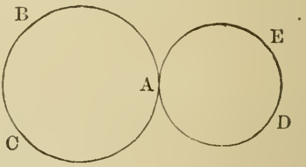


Fig. 2.



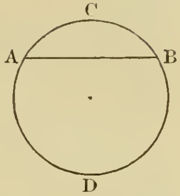
Thus the circles $ABC$, $ADE$, which meet but do not intersect, are said to touch each other. In fig. 1, the circles are said to touch one another *internally*, although in strictness only one of them touches the other internally; in fig. 2, they are said to touch one another *externally*.

6. The points at which circles touch each other, or at which straight lines touch circles, are called **points of contact.**

Thus in the figures to definitions 3 and 5, the points $A$ are points of contact.

7. A **chord** of a circle is the straight line joining any two points on the circumference.

Thus $AB$ is a chord of the circle $ABC$.



8. An **arc** of a circle is any part of the circumference.

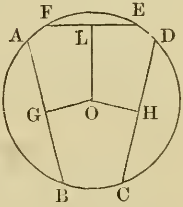
Thus $ACB$ is an arc of the circle $ABC$; so is $ADB$.

9. A chord of a circle which does not pass through the centre divides the circumference into two unequal arcs. These arcs are called the **major** and the **minor** arcs, and they are said to be **conjugate** to each other.

Thus the chord $AB$ divides the circumference of the circle $ABC$ into the conjugate arcs $ADB$, $ACB$, of which $ADB$ is a major arc, and $ACB$ a minor arc.

10. Chords of a circle are said to be **equidistant** from the centre when the perpendiculars drawn to them from the centre are equal; and one chord is farther from the centre than another, when the perpendicular on it from the centre is greater than the perpendicular on the other.

Thus in the circle $ABC$, whose centre is $O$, if the perpendiculars $OG$, $OH$ on the chords $AB$, $CD$ are equal, $AB$ and $CD$ are said to be equidistant from $O$; if the perpendicular $OL$ on the chord $EF$ is greater than $OG$ or $OH$, the chord $EF$ is said to be farther from the centre than $AB$ or $CD$.

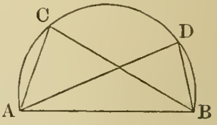


11. A **segment** of a circle is the figure contained by a chord, and either of the arcs into which the chord divides the circumference. The segments are called **major** or **minor segments**, according as their arcs are major or minor arcs.

Thus (see figure to definition 7) the figure contained by the minor arc $ACB$ and the chord $AB$ is a minor segment; the figure contained by the major arc $ADB$ and the chord $AB$ is a major segment.

It is worthy of observation that a segment, like a circle, is generally named by three letters; but the letters may not be arranged anyhow. The letters at the ends of the chord must be placed either first or last.

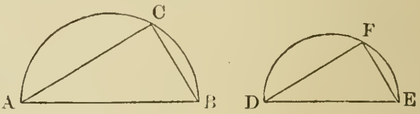
12. An **angle in a segment** of a circle is the angle contained by two straight lines drawn from any point in the arc of the segment to the ends of the chord.



Thus $ACB$ and $ADB$ are angles in the segment $ACB$.

13. **Similar segments** of circles are those which contain equal angles.

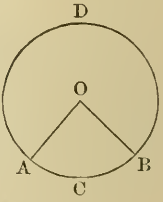
Thus if the angles $C$ and $F$ are equal, the segment $ACB$ is said to be similar to the segment $DFE$.



14. A **sector** of a circle is the figure contained by an arc and the two radii drawn to the ends of the arc.

Thus if $O$ be the centre of the circle $ABD$, the figure $OACB$ is a sector; so is $OADB$.

It is obvious that, when the radii are in the same straight line, the sector becomes a semicircle.



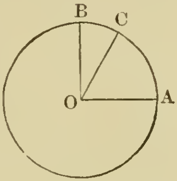
15. The **angle of a sector** is the angle contained by the two radii.

Thus the angle of the sector $OACB$ is the angle $AOB$.

16. Two radii of a circle not in the same straight line divide the circle into two sectors, one of which is greater and the other less than a semicircle; the former may be called a **major**, and the latter a **minor sector**.

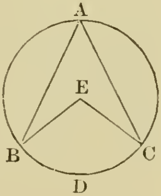
Thus $OADB$ is a major sector, and $OACB$ is a minor sector.

17. Sectors have received particular names according to the size of the angle contained by the radii. When the contained angle is a right angle, the sector is called a **quadrant**; when the contained angle is equal to one of the angles of an equilateral triangle, the sector is called a **sextant.**



Thus if $AOB$ is a right angle, or one-fourth of four right angles, the sector $OAB$ is a quadrant; if $AOC$ is two-thirds of one right angle (see p. 71, deduction 9), or one-sixth of four right angles, the sector $OAC$ is a sextant.

18. An angle is said to be at the centre, or at the circumference of a circle, when its vertex is at the centre, or on the circumference of the circle.



Thus $BEC$ is an angle at the centre, and $BAC$ an angle at the circumference of the circle $ABC$.

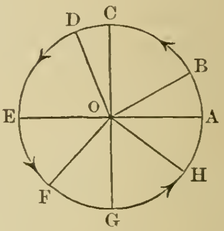
19. An angle either at the centre or at the circumference of a circle is said to stand on the arc intercepted between the arms of the angle.

Thus the angle $BEC$ at the centre and the angle $BAC$ at the circumference both stand on the same arc $BDC$.

In respect to the angle $BEC$ at the centre of the circle $ABC$, it may readily occur to the reader to inquire whether the minor arc $BDC$ is the only arc intercepted by $EB$ and $EC$, the arms of the angle. Obviously enough $EB$ and $EC$ intercept also the major arc $BAC$. What, then, is the angle which stands on the major arc $BAC$? This inquiry leads us naturally to reconsider our definition of an angle.

20. An angle may be regarded as generated (or described) by a straight line which revolves round one of its end points, the size of the angle depending on the amount of revolution.

Thus if the straight line $OB$ occupy at first the position $OA$, and then revolve round $O$ in a manner opposite to that of the hands of a watch, till it comes into the position $OB$, it will have generated or described the angle $AOB$. If $OB$ continue its revolution round $O$ till it occupies the position $OD$, it will have generated the angle $AOD$; if $OB$ still continue its revolution round $O$ till it occupies successively the positions $OF$, $OH$, it will have generated the angles $AOF$, $AOH$. The angles $AOB$, $AOD$, $AOF$, $AOH$, being successively generated by the revolution of $OB$, are therefore arranged in order of magnitude, $AOD$ being greater than $AOB$, $AOF$ greater than $AOD$, and $AOH$ greater than $AOF$.

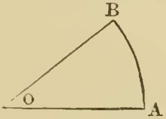


It is plain enough that $OB$, after reaching the position $OH$, may continue its revolution till it occupies the position it started from, when it will coincide again with $OA$. $OB$ will then have described a complete revolution. If the revolution be supposed to continue, the angle generated by $OB$ will grow greater and greater (since its size depends on the amount of revolution), but $OB$ itself will return to the positions it occupied before; and therefore in its second revolution $OB$ will not indicate any new direction relatively to $OA$, which it did not indicate in its first. Hence there is no need at present to consider angles greater than those generated by a straight line in one complete revolution.

21. In the course of the revolution of $OB$ from the position of $OA$ round to $OA$ again, $OB$ will at some time or other occupy the position $OE$, which is in a straight line with $OA$; the angle $AOE$ thus generated is called a **straight** (or sometimes a **flat**) **angle.**

When $OB$ occupies the position $OC$ midway between that of $OA$ and $OE$ (that is, when the angles $AOC$ and $COE$ are equal), the angle $AOC$ thus generated is called a right angle. Hence a straight angle is equal to two right angles.

When $OB$ occupies the position $OG$ which is in a straight line with $OC$, the angle $AOG$ thus generated is an angle of three right angles; when $OB$ again coincides with $OA$, it has generated an angle of four right angles. Hence angle *AOB* is less than a right angle; angle *AOD* is greater than one right angle, and less than two; angle *AOF* is greater than two and less than three right angles; angle *AOH* is greater than three and less than four right angles.



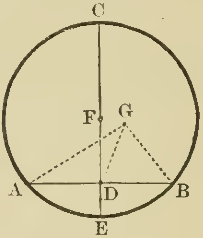
22. It has been explained how *OB*, starting from the position *OA*, and revolving in a manner opposite to that of the hands of a watch, generates the angle *AOB*, less than a right angle when it reaches the position *OB*. But we may suppose that *OB*, starting from *OA*, reaches the position *OB* by revolving round *O* in the same manner as the hands of a watch; it will then have generated another angle *AOB*, greater than three right angles. Thus it appears that two straight lines drawn from a point contain two angles having common arms and a common vertex. Such angles are said to be **conjugate**, the greater being called the **major conjugate**, and the less the **minor conjugate** angle. When, however, the angle contained by two straight lines is spoken of, the minor conjugate angle is understood to be meant.

23. It will be apparent from the preceding that the sum of two conjugate angles is equal to four right angles; and that when two conjugate angles are unequal, the minor conjugate must be less than two right angles, and the major conjugate greater than two right angles. When two conjugate angles are equal, each of them must be a straight angle.

Major conjugate angles are often called **reflex** angles, and to prevent obtuse angles from being confounded with reflex angles, obtuse angles may now be defined to be angles greater than one right angle, and less than two right angles.

## PROPOSITION 1. PROBLEM.

*To find the centre of a given circle.*



Let $ABC$ be the given circle:
*it is required to find its centre.*

Draw any chord $AB$, and bisect it at $D$; *I.* 10
from $D$ draw $DC \perp AB$, *I.* 11
and let $DC$, produced if necessary, meet the $\bigcirc^{ce}$ at $C$ and $E$.
Bisect $CE$ at $F$. *I.* 10

$F$ is the centre of $\odot$ $ABC$.

For if $F$ be not the centre, let $G$ be the centre;
and join $GA$, $GD$, $GB$.

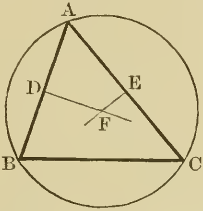
$$\text{In }\triangle\text{s } ADG, BDG, \left\{\begin{array}{ll} AD = BD & \text{Const.} \\ DG = DG & \\ GA = GB; & \text{III. Def. 1}\end{array}\right.$$

$\therefore\ \angle ADG = \angle BDG$; *I.* 8
$\therefore\ \angle ADG$ is right. *I. Def.* 10
But $\angle ADC$ is right; *Const.*
$\therefore\ \angle ADG = \angle ADC$, which is impossible;
$\therefore\ G$ is not the centre.

Now $G$ is any point out of $CE$;
$\therefore$ the centre is in $CE$.
But, since the centre is in $CE$, it must be at $F$, the middle point of $CE$.

Cor. 1.—The straight line which bisects any chord of a circle perpendicularly, passes through the centre of the circle.

Cor. 2.—Hence a circle may be described which shall pass through the three vertices of a triangle.



For if a circle could be described to pass through *A*, *B*, *C*, the vertices of the triangle *ABC*, *AB* and *AC* would be chords of this circle;

∴ *DF*, which bisects *AB* perpendicularly, would pass through the centre. *III.* 1, *Cor.* 1

Similarly *EF*, which bisects *AC* perpendicularly, would pass through the centre. *III.* 1, *Cor.* 1

Hence *F* will be the centre, and *FA*, *FB*, or *FC* the radius.

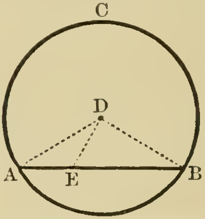
1. Show how, by twice applying Cor. 1, to find the centre of a given circle.
2. Similarly, show how to find the centre of a circle, an arc only of which is given.
3. Describe a circle to pass through three given points. When is this impossible?
4. Describe a circle to pass through two given points, and have its centre in a given straight line. When is this impossible?
5. Describe a circle to pass through two given points, and have its radius equal to a given straight line. When is this impossible?
6. A quadrilateral has its vertices situated on the ○ce of a circle. Prove that the straight lines which bisect the sides perpendicularly are concurrent.
7. From a point outside a circle two equal straight lines are drawn to the ○ce. Prove that the bisector of the angle they contain passes through the centre of the circle.

8. Show also that the same thing is true when the point is taken either within the circle or on the $\bigcirc^{ce}$.
9. Hence give another method of finding the centre of a given circle.

---

## PROPOSITION 2. Theorem.

*If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.**



Let *ABC* be a circle, *A* and *B* any two points in the $\bigcirc^{ce}$: *it is required to prove that AB shall fall within the circle.*

Find *D* the centre of the ⊙ *ABC;* *III.* 1
take any point *E* in *AB*, and join *DA*, *DE*, *DB*.

Because *DA* = *DB*, ∴ ∠ *A* = ∠ *B*. *I.* 5
But ∠ *DEB* is greater than ∠ *A ;* *I.* 16
∴ ∠ *DEB* is greater than ∠ *B;*
∴ *DB* is greater than *DE*. *I.* 19
Now since *DE* drawn from the centre of the ⊙ *ABC* is less than a radius, *E* must be within the circle. *III. Def.* 1, *Cor.* 1

But *E* is any point in *AB*, except the end points *A* and *B;*
∴ *AB* itself is within the circle.

1. Prove that a straight line cannot cut the $\bigcirc^{ce}$ of a circle in more than two points.

---

* Euclid's proof is indirect. The one in the text is found in *Clavii Commentaria in Euclidis Elementa* (1612), p. 109.

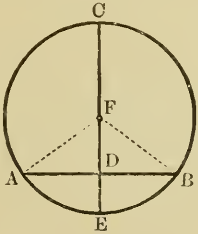
2. Describe a circle whose ○ce shall pass through a given point, whose centre shall be in one given straight line, and whose radius shall be equal to another given straight line. May more than one circle be so drawn? If so, how many? When will there be only one, and when none at all?

---

## PROPOSITION 3. Theorems.

*If a straight line drawn through the centre of a circle bisect a chord which does not pass through the centre, it shall cut it at right angles.*

*Conversely: If it cut it at right angles, it shall bisect it.*



(1) Let $ABC$ be a circle, $F$ its centre; and let $CE$, which passes through $F$, bisect the chord $AB$ which does not pass through $F$:

*it is required to prove* $CE \perp AB$.

Join $FA$, $FB$.

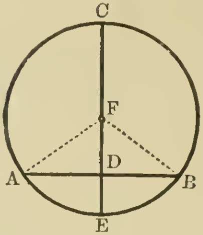
$$\text{In } \triangle\text{s } ADF, BDF, \left\{ \begin{array}{ll} AD = BD & \qquad\qquad \textit{Hyp.} \\ DF = DF & \\ FA = FB; & \qquad\qquad \textit{III. Def.}\ 1 \end{array} \right.$$

$\therefore \angle ADF = \angle BDF;$ *I.* 8

$\therefore CE$ is $\perp AB$. *I. Def.* 10

(2) In $\odot ABC$ let $CE$ be $\perp AB$:

*it is required to prove* $AD = BD$.



Join $FA$, $FB$.

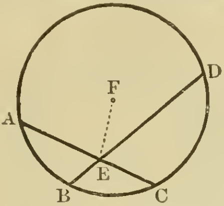
$$\text{In } \triangle\text{s } ADF, BDF, \begin{cases} \angle ADF = \angle BDF & \textit{Hyp.} \\ \angle FAD = \angle FBD & \textit{I. } 5 \\ DF = DF; & \end{cases}$$

$\therefore AD = BD.$ *I.* 26

1. In the figure to the proposition, $C$ and $E$ are on the ○$^{ce}$. Need they be so?
2. The ○$^{ce}$ of a circle passes through the vertices of a triangle. Prove that the straight lines drawn from the centre of the circle perpendicular to the sides will bisect those sides.
3. Two concentric circles intercept between their ○$^{ces}$ two equal portions of a straight line cutting them both.
4. Through a given point within a circle draw a chord which shall be bisected at that point.
5. If two chords in a circle be parallel, their middle points will lie on the same diameter.
6. Hence give a method of finding the centre of a given circle.
7. If the vertex of an isosceles triangle be taken as centre, and a circle be described cutting the base or the base produced, the segments of the base intercepted between the ○$^{ce}$ and the ends of the base will be equal.
8. If two circles cut each other, any two parallel straight lines drawn through the points of intersection to the ○$^{ces}$ will be equal.
9. If two circles cut each other, any two straight lines drawn through one of the points of intersection to the ○$^{ces}$ and making equal angles with the line of centres will be equal.

## PROPOSITION 4. THEOREM.

*If two chords of a circle cut one another and do not both pass through the centre, they do not bisect one another.*



Let $ABC$ be a circle, $AC$, $BD$ two chords which cut one another at $E$, but do not both pass through the centre: *it is required to prove that* $AC$, $BD$ *do not bisect one another.*

(1) If one of them pass through the centre, it may bisect the other which does not pass through the centre; but it cannot be itself bisected by that other.

(2) If neither of them pass through the centre, let $AE = EC$, and $BE = ED$.

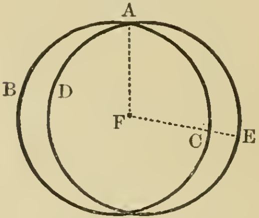
Find $F$ the centre of $\odot\ ABC$, *III.* 1
and join $FE$.

Because $FE$ passes through the centre, and bisects $AC$,
$\therefore \angle\ FEA$ is right. *III.* 3
Because $FE$ passes through the centre, and bisects $BD$,
$\therefore \angle\ FEB$ is right; *III.* 3
$\therefore \angle\ FEA = \angle\ FEB$, which is impossible.
$\therefore AC$, $BD$ do not bisect one another.

1. If two chords of a circle bisect each other, what must both of them be?
2. No $\|^{m}$ whose diagonals are unequal can have its vertices on the $\bigcirc^{ce}$ of a circle.
3. No $\|^{m}$ except a rectangle can have its vertices on the $\bigcirc^{ce}$ of a circle.

## PROPOSITION 5. THEOREM.

*If two circles cut one another, they cannot have the same centre.*



Let the ⊙s $ABC$, $ADE$ cut one another at $A$:
*it is required to prove that they cannot have the same centre.*

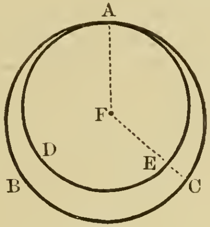
If they can, let $F$ be the common centre.
Join $FA$, and draw any other straight line $FCE$ to meet the two ○$^{\text{ces}}$.

Then $FA = FC$, being radii of ⊙ $ABC$, *III. Def.* 1
and $FA = FE$, being radii of ⊙ $ADE$; *III. Def.* 1
∴ $FC = FE$, which is impossible.
∴ ⊙s $ABC$, $ADE$ cannot have the same centre.

1. If two circles do not cut one another, can they have the same centre?
2. If two circles cut one another, can their common chord be a diameter of either of them? Can it be a diameter of both?
3. If the common chord of two intersecting circles is the diameter of one of them, prove that it is ⊥ the straight line joining the centres.
4. If two circles cut one another, the distance between their centres is less than the sum, and greater than the difference of their radii.
5. Prove the converse of the preceding deduction.

## PROPOSITION 6. THEOREM.

*If two circles touch one another internally, they cannot have the same centre.*



Let the ⊙s $ABC$, $ADE$ touch one another internally at $A$: *it is required to prove that they cannot have the same centre.*

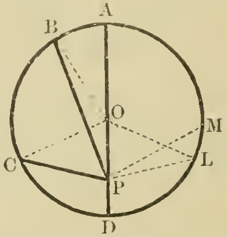
If they can, let $F$ be the common centre.
Join $FA$, and draw any other straight line $FEC$ to meet the two $\bigcirc^{\text{ces}}$.

Then $FA = FC$, being radii of ⊙ $ABC$, *III. Def.* 1
and $FA = FE$, being radii of ⊙ $ADE$; *III. Def.* 1
∴ $FC = FE$, which is impossible.
∴ ⊙s $ABC$, $ADE$ cannot have the same centre.

1. If two circles touch one another externally, can they have the same centre?
2. Enunciate III. 5, 6, and the preceding deduction in one statement.
3. If one circle be inside another, and do not touch it, the distance between their centres is less than the difference of their radii.
4. If one circle be outside another and do not touch it, the distance between their centres is greater than the sum of their radii.
5. Prove the converses of the two preceding deductions.

## PROPOSITION 7. THEOREM.

*If from any point within a circle which is not the centre, straight lines be drawn to the circumference, the greatest is that which passes through the centre, and the remaining part of that diameter is the least; of the others, that which is nearer to the greatest is greater than the more remote; and from the given point straight lines which are equal to one another can be drawn to the circumference only in pairs, one on each side of the diameter.*



Let $ABC$ be a circle, and $P$ any point within it which is not the centre; from $P$ let there be drawn to the $\bigcirc^{ce}$ $DPA$, $PB$, $PC$, of which $DPA$ passes through the centre $O$:

*it is required to prove* (1) *that $PA$ is greater than $PB$;*
(2) *that $PB$ is greater than $PC$;*
(3) *that $PD$ is less than $PC$;*
(4) *that only one straight line can be drawn from $P$ to the $\bigcirc^{ce}$ = $PC$.*

Join $OB$, $OC$.

(1) Because $OB = OA$, being radii of the same circle;
$\therefore$ $PO + OB = PO + OA$, or $PA$.
But $PO + OB$ is greater than $PB$; I. 20
$\therefore$ $PA$ is greater than $PB$.

(2) In $\triangle$s $POB$, $POC$,

$$\begin{cases} PO = PO & \\ OB = OC & \textit{III. Def. } 1 \\ \angle\, POB \text{ is greater than } \angle\, POC; & \end{cases}$$

$\therefore$ $PB$ is greater than $PC$. *I.* 24

(3) Because $OC - OP$ is less than $PC$, *I.* 20, *Cor.*
and $OC = OD$, being radii of the same circle;
$\therefore$ $OD - OP$ is less than $PC$;
$\therefore$ $PD$ is less than $PC$.

(4) At $O$ make $\angle\, POL = \angle\, POC$, *I.* 23
and join $PL$.

In $\triangle$s $POL$, $POC$,

$$\begin{cases} PO = PO & \\ OL = OC & \textit{III. Def. } 1 \\ \angle\, POL = \angle\, POC; & \textit{Const.} \end{cases}$$

$\therefore$ $PL = PC$. *I.* 4
And besides $PL$ no other straight line can be drawn from $P$ to the $\bigcirc^{ce} = PC$.
For if $PM$ were also $= PC$,
then $PM = PL$, which is impossible.

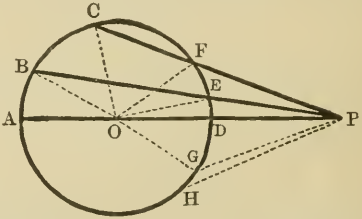
Cor.—If from a point inside a circle more than two equal straight lines can be drawn to the $\bigcirc^{ce}$, that point must be the centre.

For another proof of this Cor., see III. 9.

1. Prove $PC$ greater than $PD$, using I. 20 instead of I. 20, Cor.
2. Wherever the point $P$ be taken, provided it be inside the circle $ABC$, the sum of the greatest and the least straight lines that can be drawn from it to the $\bigcirc^{ce}$ is constant.
3. Find another point whose greatest and least distances from the $\bigcirc^{ce}$ are respectively = those of $P$ from the $\bigcirc^{ce}$. How many such points are there? Where do they lie?
4. Prove, by considering $POA$ and $POD$ as infinitely thin triangles, that $PA$ is greater than $PB$, and $PC$ greater than $PD$ by I. 24.

## PROPOSITION 8. THEOREM.

*If from any point without a circle straight lines be drawn to the circumference, of those which fall upon the concave part of the circumference the greatest is that which passes through the centre, and of the others that which is nearer to the greatest is greater than the more remote: but of those which fall on the convex part of the circumference the least is that which, when produced, passes through the centre, and of the others that which is nearer to the least is less than the more remote; and from the given point straight lines which are equal to one another can be drawn to the circumference only in pairs, one on each side of the diameter.*



Let *ABC* be a circle, and *P* any point without it; from *P* let there be drawn to the $\bigcirc^{ce}$ *PDA*, *PEB*, *PFC*, of which *PDA* passes through the centre *O*:

*it is required to prove* (1) *that PA is greater than PB;*

(2) *that PB is greater than PC;*

(3) *that PD is less than PE;*

(4) *that PE is less than PF;*

(5) *that only one straight line can be drawn from P to the* $\bigcirc^{ce} = PF$.

Join *OB*, *OC*, *OE*, *OF*.

(1) Because $OB = OA$, being radii of the same circle;

$\therefore$ $PO + OB = PO + OA$, or *PA*.

But $PO + OB$ is greater than *PB*; *I.* 20

$\therefore$ *PA* is greater than *PB*.

(2) In $\triangle$s $POB$, $POC$, $\left\{\begin{array}{ll} PO = PO & \\ OB = OC & \textit{III. Def.}\ 1 \\ \angle POB \text{ is greater than } \angle POC; & \end{array}\right.$

$\therefore$ $PB$ is greater than $PC$. *I.* 24

(3) Because $OP - OE$ is less than $PE$, *I.* 20, *Cor.*
and $OE = OD$, being radii of the same circle;
$\therefore$ $OP - OD$ is less than $PE$;
$\therefore$ $PD$ is less than $PE$.

(4) In $\triangle$s $POE$, $POF$, $\left\{\begin{array}{ll} PO = PO & \\ OE = OF & \textit{III. Def.}\ 1 \\ \angle POE \text{ is less than } \angle POF; & \end{array}\right.$

$\therefore$ $PE$ is less than $PF$. *I.* 24

(5) At $O$ make $\angle POG = \angle POF$, *I.* 23
and join $PG$.

In $\triangle$s $POG$, $POF$, $\left\{\begin{array}{ll} PO = PO & \\ OG = OF & \textit{III. Def.}\ 1 \\ \angle POG = \angle POF; & \textit{Const.} \end{array}\right.$

$\therefore$ $PG = PF$. *I.* 4

And besides $PG$ no other straight line can be drawn from $P$ to the $\bigcirc^{ce} = PF$.
For if $PH$ were also $= PF$,
then $PH = PG$, which is impossible.

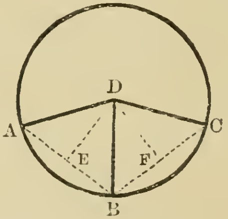
1. Prove $PE$ greater than $PD$, using I. 20 instead of I. 20, Cor.
2. Prove that $PE$ is less than $PF$, using I. 21 instead of I. 24.
3. Wherever the point $P$ be taken, provided it be outside the circle $ABC$, the difference of the greatest and the least straight lines that can be drawn from it to the $\bigcirc^{ce}$ is constant.
4. Compare the enunciations of the last deduction and of the analogous one from III. 7, and state and prove the corresponding theorem when the point $P$ is on the $\bigcirc^{ce}$ of the $\odot$ $ABC$.
5. Prove that $AD$ is greater than $BE$, and $BE$ greater than $CF$.
6. If the straight line $PFC$ be supposed to revolve round $P$ as a pivot, till the points $F$ and $C$ coincide, what would the straight line $PFC$ become?
7. The tangent to a circle from any external point is less than any secant to the circle from that point, and greater than the external segment of the secant.

8. Could a line be drawn to separate the concave from the convex part of the $\bigcirc^{ce}$ of the $\odot$ *ABC* viewed from the point *P?* How?

---

## PROPOSITION 9. THEOREM.

*If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre.*[*]



Let *ABC* be a circle, and let three equal straight lines *DA*, *DB*, *DC* be drawn from the point *D* to the $\bigcirc^{ce}$: *it is required to prove that D is the centre of the circle.*

Join *AB*, *BC*, and bisect them at *E*, *F*; *I.* 10
and join *DE*, *DF*.

$$\text{In } \triangle\text{s } AED, BED, \left\{\begin{array}{l} AE = BE \quad \textit{Const.} \\ ED = ED \\ DA = DB; \quad \textit{Hyp.} \end{array}\right.$$

$\therefore \angle AED = \angle BED$; *I.* 8

$\therefore$ *DE* is $\perp$ *AB*;

$\therefore$ *DE*, since it bisects *AB* perpendicularly, must pass through the centre of the circle. *III.* 1, *Cor.* 1

Hence also *DF* must pass through the centre;

$\therefore$ *D*, the only point common to *DE* and *DF*, is the centre.

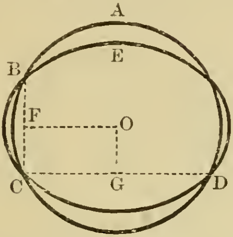
Prove the proposition by using the eighth deduction from III. 1.

---

[*] In the MSS. of Euclid, two proofs of this proposition occur, only the second of which Simson inserted in his edition. The one given in the text is the first.

## PROPOSITION 10. THEOREM.

*One circle cannot cut another at more than two points.*[*]



If it be possible, let the $\odot$ $ABC$ cut the $\odot$ $EBC$ at more than two points—namely, at $B$, $C$, $D$.

Join $BC$, $CD$, and bisect them at $F$ and $G$; *I.* 10
through $F$ and $G$ draw $FO$, $GO \perp BC$, $CD$, *I.* 11
and let $FO$, $GO$ intersect at $O$.

Because $BC$ is a chord in both circles, and $FO$ bisects it perpendicularly,
$\therefore$ the centres of both circles lie in $FO$. *III.* 1, *Cor.* 1
Hence also the centres of both circles lie in $GO$;
$\therefore$ $O$ is the centre of both circles,
which is impossible, since they cut one another. *III.* 5
$\therefore$ one circle cannot cut another at more than two points.

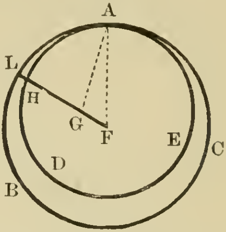
1. Two circles cannot meet each other in more than two points.
2. If two circles have three points in common, how must they be situated?
3. Show, by supposing the radius of one of the circles to increase indefinitely in length, that the first deduction from III. 2 is a particular case of this proposition.

---

[*] In the MSS. of Euclid, two proofs of this proposition occur, only the second of which Simson inserted in his edition. The one given in the text is the first.

## PROPOSITION 11. THEOREM.

*If two circles touch one another internally at any point, the straight line which joins their centres, being produced, shall pass through that point.*



Let the two $\odot$s *ABC*, *ADE*, whose centres are *F* and *G*, touch one another internally at the point *A*:
*it is required to prove that FG produced passes through A.*

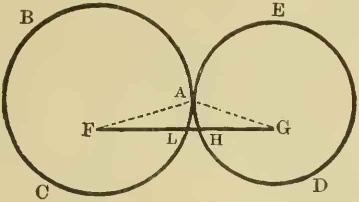
If not, let it pass otherwise, as *FGHL*.
Join *FA*, *GA*.

Because $FA = FL$, being radii of $\odot$ *ABC*, *III. Def.* 1
and $GA = GH$, being radii of $\odot$ *ADE*; *III. Def.* 1
$\therefore FA - GA = FL - GH,$
$= FG + HL;$
$\therefore FA - GA$ is greater than *FG* by *HL*.
But $FA - GA$ is less than *FG*; *I.* 20, *Cor.*
$\therefore FA - GA$ is both greater and less than *FG*, which is impossible;
$\therefore$ *FG* produced must pass through *A*.

1. If two circles touch internally, the distance between their centres is equal to the difference of their radii.
2. Two circles touch internally at a point, and through that point a straight line is drawn to cut the $\circ^{ces}$ of the two circles. If the points of intersection be joined with the respective centres, the two straight lines will be parallel.
3. This proposition is a particular case of the tenth deduction from I. 8.

## PROPOSITION 12. Theorem.

*If two circles touch one another externally at any point, the straight line which joins their centres shall pass through that point.*



Let the two ⊙s $ABC$, $ADE$, whose centres are $F$ and $G$, touch one another externally at the point $A$:
*it is required to prove that $FG$ passes through $A$.*

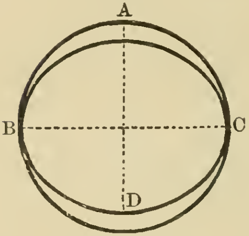
If not, let it pass otherwise, as $FLHG$.
Join $FA$, $GA$.

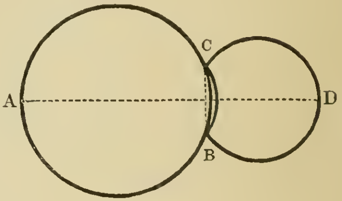
Because $FA = FL$, being radii of ⊙ $ABC$, *III. Def.* 1
and $GA = GH$, being radii of ⊙ $ADE$; *III. Def.* 1
$\therefore FA + GA = FL + GH,$
$= FG - HL;$
$\therefore FA + GA$ is less than $FG$ by $HL$.
But $FA + GA$ is greater than $FG$; *I.* 20
$\therefore FA + GA$ is both less and greater than $FG$, which is impossible;
$\therefore FG$ must pass through $A$.

1. If two circles touch externally, the distance between their centres is equal to the sum of their radii.
2. Two circles touch externally at a point, and through that point a straight line is drawn to cut the ○$^{ces}$ of the two circles. If the points of intersection be joined with the respective centres, the two straight lines will be parallel.
3. This proposition is a particular case of the tenth deduction from I. 8.

## PROPOSITION 13. THEOREM.

*Two circles cannot touch each other at more points than one, whether internally or externally.*





For, if it be possible, let the two ⊙s *ABC*, *BDC* touch each other at the points *B* and *C*.

Join *BC*, and draw *AD* bisecting *BC* perpendicularly. *I.* 10, 11

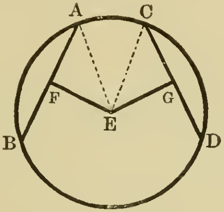
Because *B* and *C* are points in the $\odot^{ces}$ of both circles,
∴ *BC* is a chord of both circles.
And because *AD* bisects *BC* perpendicularly, *Const.*
∴ *AD* passes through the centres of both circles; *III.* 1, *Cor.* 1.
∴ *AD* passes also through the points of contact *B* and *C*, *III.* 11, 12
which is impossible.
Hence the two ⊙s *ABC*, *BDC* cannot touch each other at more points than one, whether internally or externally.

1. If the distance between the centres of two circles be equal to the sum of their radii, the two circles touch each other externally.
2. If the distance between the centres of two circles be equal to the difference of their radii, the two circles touch each other internally.

## PROPOSITION 14. THEOREMS.

*Equal chords in a circle are equidistant from the centre.*

*Conversely: Chords in a circle which are equidistant from the centre are equal.*



(1) Let $AB$, $CD$ be equal chords in the $\odot$ $ABC$, and $EF$, $EG$ their distances from the centre $E$:
*it is required to prove* $EF = EG$.

Join $EA$, $EC$.

Because $EF$ drawn through the centre $E$ is $\perp$ $AB$,
$\therefore$ $EF$ bisects $AB$, that is, $AB$ is double of $AF$. *III.* 3
Hence also $CD$ is double of $CG$.
Now since $AB = CD$, $\therefore$ $AF = CG$, and $AF^2 = CG^2$.
But because $EA = EC$, $\therefore$ $EA^2 = EC^2$;
$\therefore$ $AF^2 + FE^2 = CG^2 + GE^2$. *I.* 47
Take away $AF^2$ and $CG^2$ which are equal;
$\therefore$ $FE^2 = GE^2$, and $FE = GE$.

(2) Let $AB$, $CD$ be chords in the $\odot$ $ABC$, and let $EF$, $EG$, their distances from the centre $E$, be equal:
*it is required to prove* $AB = CD$.

Join $EA$, $EC$.

It may be proved as before that $AB = 2\,AF$, $CD = 2\,CG$, and that $AF^2 + FE^2 = CG^2 + GE^2$.

Now $FE^2 = GE^2$, since $FE = GE$; *Hyp.*

$\therefore$ $AF^2 = CG^2$, and $AF = CG$;

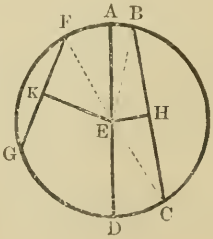
$\therefore$ $2\,AF = 2\,CG$, that is, $AB = CD$.

1. If a series of equal chords be placed in a circle, their middle points will lie on the ○ᶜᵉ of another circle.
2. Two parallel chords in a circle whose diameter is 10 inches, are 8 inches and 6 inches; find the distance between them.
3. If two chords of a circle intersect each other and make equal angles with the diameter drawn through their point of intersection, they are equal.
4. If two secants of a circle intersect, and make equal angles with the diameter drawn through their point of intersection, those parts of the secants intercepted by the ○ᶜᵉ are equal.
5. If in a given circle a chord of given length be placed, the distance of the chord from the centre will be fixed.
6. Prove the converse of the preceding deduction.
7. If two equal chords intersect either within or without a circle, the segments of the one are equal to the segments of the other.

## PROPOSITION 15. THEOREMS.

*The diameter is the greatest chord in a circle; and of all others that which is nearer to the centre is greater than one more remote.*

*Conversely: The greater chord is nearer to the centre than the less.*



Let $ABC$ be a circle of which $AD$ is a diameter, and $BC$, $FG$ two other chords whose distances from the centre $E$ are $EH$, $EK$:

*it is required to prove:*

(1) *that $AD$ is greater than $BC$ or $FG$;*

(2) *that, if $EH$ is less than $EK$, $BC$ must be greater than $FG$;*

(3) *that, if $BC$ is greater than $FG$, $EH$ must be less than $EK$.*

(1) Join $EB$, $EC$.

Because $AE = BE$, and $ED = EC$; *III. Def.* 1
$\therefore AD = BE + EC$.
But $BE + EC$ is greater than $BC$; *I.* 20
$\therefore AD$ is greater than $BC$.

(2) Join $EB$, $EC$, $EF$.

It may be proved, as in the preceding proposition,
that $BC$ is double of $BH$, that $FG$ is double of $FK$,
and that $EH^2 + HB^2 = EK^2 + KF^2$.
Now $EH^2$ is less than $EK^2$, since $EH$ is less than $EK$; *Hyp.*
$\therefore HB^2$ is greater than $KF^2$, and $HB$ greater than $KF$.
$\therefore$ twice $HB$ is greater than twice $KF$,
that is, $BC$ is greater than $FG$.

(3) Join $EB$, $EC$, $EF$.

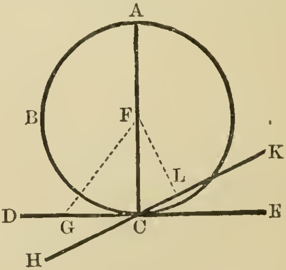
It may be proved, as before, that $BC = 2\,BH$, $FG = 2\,FK$,
and that $EH^2 + HB^2 = EK^2 + KF^2$.
Now, since $BC$ is greater than $FG$, *Hyp.*
$\therefore BH$ is greater than $FK$, and $BH^2$ greater than $FK^2$.
Hence $EH^2$ must be less than $EK^2$, and $EH$ less than $EK$

1. **The shortest chord that can be drawn through a given point within a circle is that which is perpendicular to the diameter through the point.**

2. Of two chords of a circle which intersect each other, and make unequal angles with the diameter drawn through their point of intersection, that which makes the less angle is the greater.
3. If two secants of a circle intersect each other, and make unequal angles with the diameter drawn through their point of intersection, that part which is intercepted by the ○$^{ce}$ on the secant making the less angle is greater than the corresponding part on the other.
4. Through either of the points of intersection of two circles draw the greatest possible straight line terminated both ways by the ○$^{ces}$. Draw also the least possible, and show that the two are at right angles to each other.

## PROPOSITION 16. Theorem.

*The straight line drawn perpendicular to a diameter of a circle from either end of it, is a tangent to the circle; and every other straight line drawn through the same point cuts the circle.**



---

* Euclid's proof of this proposition is indirect. The one in the text is given by Orontius Finæus (1544), the second part, however, being somewhat simplified.

Let $ABC$ be a circle, of which $F$ is the centre and $AC$ a diameter; through $C$ let there be drawn $DE \perp AC$, and any other straight line $HK$:
*it is required to prove that* $DE$ *is a tangent to the* ⊙ $ABC$, *and that* $HK$ *cuts the circle.*

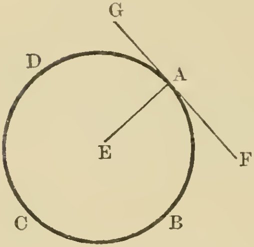
Take any point $G$ in $DE$, and join $FG$;
from $F$ draw $FL \perp HK$. *I.* 12

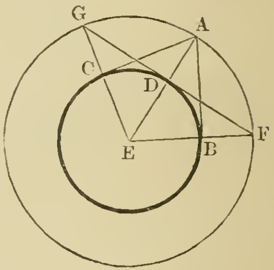
Because $\angle FCG$ is right, *Hyp.*
$\therefore$ $FG$ is greater than $FC$, a radius of the circle; *I.* 19 *Cor.*
$\therefore$ the point $G$ must be outside the circle. *III. Def.* 1, *Cor.* 2
Now $G$ is any point in $DE$, except the point $C$;
$\therefore$ $DE$ is a tangent to the circle. *III. Def.* 3
Again, because $\angle FLC$ is right, *Const.*
$\therefore$ $FL$ is less than $FC$, a radius of the circle; *I.* 19 *Cor.*
$\therefore$ the point $L$ must be inside the circle. *III. Def.* 1, *Cor.* 2
Now $L$ is a point in $HK$;
$\therefore$ $HK$ cuts the circle.

1. Draw a tangent to a circle at a given point on the ○$^{ce}$.
2. Only one tangent can be drawn to a circle at a given point on its ○$^{ce}$.
3. Two (or a series of) circles touch each other, externally or internally, at the same point. Prove that they have the same tangent at that point.
4. If a series of equal chords be placed in a circle, they will be tangents to another circle concentric with the former.
5. A straight line will cut, touch, or lie entirely outside a circle, according as its distance from the centre is less than, equal to, or greater than a radius.
6. Draw a tangent to a circle which shall be ∥ a given straight line.
7. Draw a tangent to a circle which shall be ⊥ a given straight line.
8. Draw a tangent to a circle which shall make a given angle with a given straight line. How many tangents can be drawn in each of the three cases?

## PROPOSITION 17. Problem.

*To draw a tangent to a circle from a given point.*





Let *BDC* be the given circle, and *A* the given point:
*it is required to draw a tangent to the* $\odot$ *BDC from A.*

Case 1.—When the given point *A* is inside the $\odot$ *BDC*, the problem is impossible.

Case 2.—When the given point *A* is on the $\bigcirc^{ce}$ of the $\odot$ *BDC*.

Find *E* the centre of the $\odot$ *BDC*; *III.* 1
join *EA*, and through *A* draw *FG* $\perp$ *EA*. *I.* 11
Then *FG* is a tangent to the $\odot$ *BDC*. *III.* 16

Case 3.—When the given point *A* is outside the $\odot$ *BDC*.

Find *E* the centre of the $\odot$ *BDC*: *III.* 1
and join *AE*, cutting the $\bigcirc^{ce}$ of $\odot$ *BDC* at *D*.
With centre *E* and radius *EA*, describe $\odot$ *AGF*;
through *D* draw *FDG* $\perp$ *AE*, and meeting the $\bigcirc^{ce}$ of $\odot$ *AGF* at *F* and *G*. *I.* 11
Join *EF*, *EG*, cutting the $\bigcirc^{ce}$ of $\odot$ *BDC* at *B* and *C*,
and join *AB*, *AC*. *AB* or *AC* is the required tangent.

$$\text{In } \triangle\text{s } ABE, FDE, \left\{\begin{array}{l} AE = FE \\ EB = ED \\ \angle E = \angle E; \end{array}\right. \qquad \begin{array}{r} \textit{III. Def.}\ 1 \\ \textit{III. Def.}\ 1 \\ \ \end{array}$$

$\therefore \angle ABE = \angle FDE$, *I.* 4

$= \text{a right angle.}$ *Const.*

$\therefore AB$ is a tangent to the $\odot$ $BDC$. *III.* 16

Hence also, $AC$ is a tangent to the $\odot$ $BDC$.

Cor.—The two tangents that can be drawn to a circle from an external point are equal.

By comparing $\triangle$s $ABE$, $FDE$ it may be proved that $AB = FD$; *I.* 4

and by comparing $\triangle$s $ACE$, $GDE$, it may be proved that $AC = GD$. *I.* 4

Now, since $FG$ is a chord of the $\odot$ $AFG$, and $ED$ drawn through the centre is $\perp FG$; *Const.*

$\therefore FD = GD$. *III.* 3

Hence $AB = AC$.

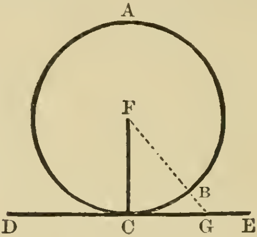
1. Prove $AB = AC$ by (*a*) I. 47, (*b*) I. 5, 6.
2. The tangents $AB$, $AC$ make equal angles with the diameter through $A$.
3. Prove $\angle BAC$ supplementary to $\angle BEC$. State this result in words.
4. No more than two tangents can be drawn to a circle from an external point.
5. If a quadrilateral be circumscribed * about a circle, the sum of two opposite sides is equal to the sum of the other two.
6. Generalise the preceding deduction.
7. If a $\|^{\text{m}}$ be circumscribed about a circle, it must be a rhombus.
8. From a point outside a circle two tangents are drawn. The straight line joining the point with the centre bisects perpendicularly the chord of contact. (In fig. 2, $BC$ is the chord of contact.)

---

* A figure is circumscribed about a circle when its sides touch the circle.

## PROPOSITION 18. THEOREM.

*The radius of a circle drawn to the point of contact of a tangent is perpendicular to the tangent.*



Let $ABC$ be a circle whose centre is $F$, and $DE$ a tangent to it at the point $C$:

*it is required to prove that the radius* $FC$ *is* $\perp$ $DE$.

If not, from $F$ draw $FG \perp DE$, and meeting the $\bigcirc^{ce}$ at $B$. *I.* 12

Because $\angle FGC$ is a right angle, *Const.*

$\therefore$ $FG$ is less than $FC$. *I.* 19 *Cor.*

But $FC = FB$; *III. Def.* 1

$\therefore$ $FG$ is less than $FB$,

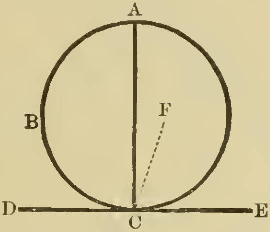
which is impossible;

$\therefore$ $FC$ must be $\perp$ $DE$.

1. Tangents at the ends of a diameter of a circle are parallel.
2. If a series of chords in a circle be tangents to another concentric circle, the chords are all equal.
3. If two circles be concentric, and a chord of the greater be a tangent to the less, it is bisected at the point of contact.
4. Through a given point within a circle draw a chord which shall be equal to a given length. May the given point be outside the circle? What are the limits to the given length?
5. Deduce this proposition from I. 5, by supposing the tangent $DE$ to be at first a secant.
6. Two circles, whose centres are $A$ and $B$, have a common tangent $CD$; prove $AC \parallel BD$.

## PROPOSITION 19. THEOREM.

*The straight line drawn from the point of contact of a tangent to a circle perpendicular to the tangent passes through the centre of the circle.*



Let $DE$ be a tangent to the $\odot\ ABC$ at the point $C$, and let $CA$ be $\perp\ DE$:
*it is required to prove that CA passes through the centre.*

If not, let $F$ be the centre,
and join $FC$.

Then $\angle\ FCE$ is right. *III.* 18.
But $\angle\ ACE$ is right; *Hyp.*
$\therefore\ \angle\ FCE = \angle\ ACE$, which is impossible;
$\therefore\ CA$ must pass through the centre of the circle.

1. In the figure, $A$ is on the $\bigcirc^{ce}$. Need it be so?
2. This proposition is a particular case of III. 1, Cor. 1.
3. A series of circles touch a given straight line at a given point. Where will their centres all lie?
4. Describe a circle to touch two given straight lines at two given points. When is this problem possible?
5. If two tangents be drawn to a circle from any point, the angle contained by the tangents is double the angle contained by the chord of contact and the diameter drawn through either point of contact.

## PROPOSITION 20. THEOREM.

*An angle at the centre of a circle is double of an angle at the circumference which stands on the same arc.*

Fig. 1.

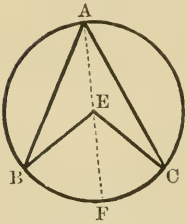


Fig. 2.

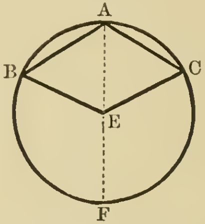
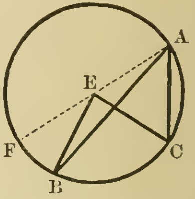


Fig. 3.



In the $\odot ABC$ let $\angle BEC$ at the centre and $\angle BAC$ at the $\bigcirc^{ce}$ stand on the same arc $BC$:
*it is required to prove* $\angle BEC =$ *twice* $\angle BAC$.

Join $AE$ and produce it to $F$.

Because $EA = EC$, $\therefore \angle EAC = \angle ECA$; I. 5

$\therefore \angle EAC + \angle ECA =$ twice $\angle EAC$.

But $\angle FEC = \angle EAC + \angle ECA$; I. 32

$\therefore \angle FEC =$ twice $\angle EAC$.

Similarly $\angle FEB =$ twice $\angle EAB$.

Hence, in figs. 1 and 2,

$\angle FEC + \angle FEB =$ twice $\angle EAC +$ twice $\angle EAB$,

that is, $\angle BEC =$ twice $\angle BAC$;

and in fig. 3,

$\angle FEC - \angle FEB =$ twice $\angle EAC -$ twice $\angle EAB$,

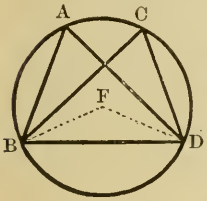
that is, $\angle BEC =$ twice $\angle BAC$.

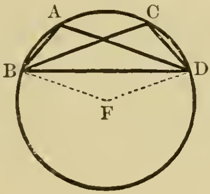
1. In the figures to the proposition, $F$ is on the $\bigcirc^{ce}$. Need it be so?
2. The angle in a semicircle is a right angle.
3. $B$ and $C$ are two fixed points in the $\bigcirc^{ce}$ of the circle $ABC$. Prove that wherever $A$ be taken on the arc $BAC$, the magnitude of the angle $BAC$ is constant.

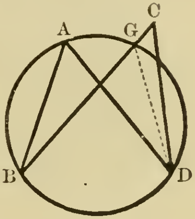
## PROPOSITION 21. THEOREMS.

*Angles in the same segment of a circle are equal.*

*Conversely: If two equal angles stand on the same arc, and the vertex of one of them be on the conjugate arc, the vertex of the other will also be on it.*[*]







(1) Let $ABD$ be a circle, and $\angle$s $A$ and $C$ in the same segment $BCD$:
*it is required to prove* $\angle A = \angle C$.

Find $F$ the centre of the $\odot\ ABD$, *III.* 1
and join $BF$, $DF$.

Then $\angle BFD =$ twice $\angle A$, *III.* 20
and $\angle BFD =$ twice $\angle C$; *III.* 20
$\therefore$ $\angle A = \angle C$.

(2) Let $\angle$s $A$ and $C$, which are equal, stand on the same arc $BD$, and let the vertex $A$ be on the conjugate arc $BAD$:
*it is required to prove that the vertex C will also be on it.*

If not, let the arc $BAD$ cut $BC$ or $BC$ produced at $G$; join $DG$.

Then $\angle A = \angle BGD$. *III.* 21
But $\angle A = \angle C$; *Hyp.*
$\therefore$ $\angle BGD = \angle C$, which is impossible. *I* 16
Hence $C$ must be on the circle which passes through $B$, $A$, $D$.

---

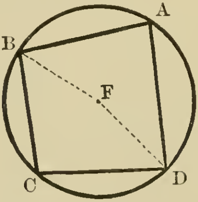
[*] The second part of this proposition is not given by Euclid.

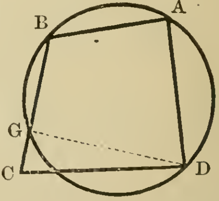
1. In the figure to III. 4, if *AB, CD* be joined, △ *AEB* is equiangular to △ *DEC.*
2. If from a point *E* outside a circle, two secants *ECA*, *EBD* be drawn, and *AB*, *CD* be joined, △ *AEB* is equiangular to △ *DEC.*
3. Given three points on the $\bigcirc^{ce}$ of a circle; find any number of other points on the $\bigcirc^{ce}$ without knowing the centre.
4. Two tangents *AB*, *AC* are drawn to a circle from an external point *A*; *D* is any point on the $\bigcirc^{ce}$ outside the △ *ABC.* Show that the sum of ∠s *ABD, ACD* is constant.
5. Is the last theorem true when *D* lies elsewhere on the $\bigcirc^{ce}$?
6. Segments of two circles stand upon a common chord *AB.* Through *C*, any point in one segment, are drawn the straight lines *ACE*, *BCD* meeting the other segment in *E*, *D*. Prove that the length of the arc *DE* is invariable wherever the point *C* be taken.

## PROPOSITION 22. THEOREMS.

*The opposite angles of a quadrilateral inscribed in a circle are supplementary.*

*Conversely: If the opposite angles of a quadrilateral be supplementary, a circle may be circumscribed about the quadrilateral.**





(1) Let the quadrilateral *ABCD* be inscribed in the ⊙ *ABC*: *it is required to prove that* $\angle A + \angle C = 2$ *rt.* ∠*s.*

Find *F* the centre of the ⊙ *ABD*, *III.* 1
and join *BF*, *DF*.

---

* The second part of this proposition is not given by Euclid, and he proves the first part by joining *AC*, *BD*.

Then $\angle BFD$ = twice $\angle A$, *III.* 20
and the reflex $\angle BFD$ = twice $\angle C$; *III.* 20
$\therefore$ the sum of the two conjugate $\angle$s $BFD$
= twice $\angle A$ + twice $\angle C$.
But the sum of the two conjugate $\angle$s $BFD$
= 4 rt. $\angle$s; *III. Def.* 23
$\therefore$ $\angle A + \angle C = 2$ rt. $\angle$s.

(2) Let $\angle$s $A$ and $C$, which are supplementary, be opposite angles of the quadrilateral $ABCD$,
and the vertex $A$ be on an arc $BAD$ which passes also through $B$ and $D$:
*it is required to prove that the vertex C will be on the conjugate arc.*

If not, let the arc conjugate to $BAD$ cut $BC$ or $BC$ produced at $G$; *III.* 1, *Cor.* 2
join $DG$.

Then $\angle A$ is supplementary to $\angle BGD$. *III.* 22
But $\angle A$ is supplementary to $\angle C$; *Hyp.*
$\therefore$ $\angle BGD = \angle C$, which is impossible. *I.* 16
Hence $C$ must be on the circle which passes through $B$, $A$, $D$.

Cor.—If one side of a quadrilateral inscribed in a circle be produced, the exterior angle is equal to the remote interior angle of the quadrilateral.

For each is supplementary to the interior adjacent angle.
*I.* 13, *III.* 22

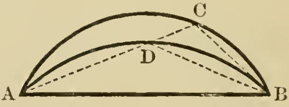
1. If a $\|^{\text{m}}$ be inscribed in a circle, it must be a rectangle.
2. If, from a point $E$ outside a circle, two secants $ECA$, $EBD$ be drawn, and $AD$, $BC$ be joined, $\triangle AED$ is equiangular to $\triangle BEC$.
3. If a polygon of an even number of sides (a hexagon, for example) be inscribed in a circle, the sum of its alternate angles is half the sum of all its angles.
4. If an arc be divided into any two parts, the sum of the angles in the two segments is constant.

5. Divide a circle into two segments, such that the angle in the one segment shall be (*a*) twice, (*b*) thrice, (*c*) five times, (*d*) seven times the angle in the other segment.
6. $ACB$ is a right-angled triangle, right-angled at $C$, and $O$ is the point of intersection of the diagonals of the square described on $AB$ outwardly to the triangle; prove that $CO$ bisects $\angle ACB$.
7. What modification must be made on the last theorem when the square is described on $AB$ inwardly to the triangle?
8. If two chords cut off one pair of similar segments from two circles, the other pair of segments they cut off are also similar.
9. Given three points on the $\bigcirc^{ce}$ of a circle: find any number of other points on the $\bigcirc^{ce}$ without knowing the centre.
10. $ABC$ is a triangle; $AX$, $BY$, $CZ$ are the three perpendiculars from the vertices on the opposite sides, intersecting at $O$. Prove the following sets of four points concyclic (that is, situated on the $\bigcirc^{ce}$ of a circle): $A, Z, O, Y$; $B, X, O, Z$; $C, Y, O, X$; $A, B, X, Y$; $B, C, Y, Z$; $C, A, Z, X$.

---

## PROPOSITION 23. THEOREM.

*On the same chord and on the same side of it there cannot be two similar segments of circles not coinciding with one another.*



If it be possible, on the same chord $AB$, and on the same side of it, let there be two similar segments of ⊙s $ACB$, $ADB$ not coinciding with one another.

Draw any straight line $ADC$ cutting the arcs of the segments at $D$ and $C$;
and join $BC$, $BD$.

Because segment $ADB$ is similar to segment $ACB$, *Hyp.*
$\therefore \angle ADB = \angle ACB$, *III. Def.* 13
which is impossible, *I.* 16

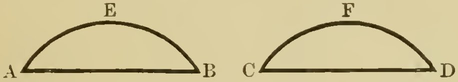
Hence two similar segments on the same chord and on the same side of it must coincide.

1. Of all the segments of circles on the same side of the same chord, that which is the greatest contains the least angle.
2. Prove by this proposition the second part of III. 21.

---

## PROPOSITION 24. THEOREM.

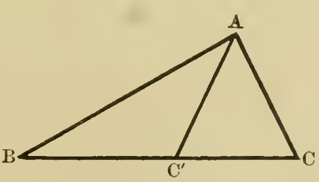
*Similar segments of circles on equal chords are equal.*



Let $AEB$, $CFD$ be similar segments on equal chords $AB$, $CD$:
*it is required to prove segment* $AEB$ = *segment* $CFD$.

If segment $AEB$ be applied to segment $CFD$,
so that $A$ falls on $C$, and so that $AB$ falls on $CD$;
then $B$ will coincide with $D$, because $AB = CD$. *Hyp.*
Hence the segment $AEB$ being similar to the segment $CFD$, must coincide with it; *III.* 23
∴ segment $AEB$ = segment $CFD$.

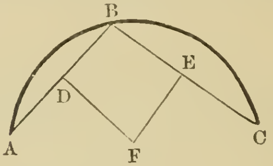
1. Similar segments of circles on equal chords are parts of equal circles.
2. $ABC$, $ABC'$ are two △s such that $AC = AC'$. Prove that the circle which passes through $A$, $B$, $C$ is equal to the circle which passes through $A$, $B$, $C'$.



3. If $ABCD$ is a ∥$^{m}$, and $BE$ makes with $AB$, ∠ $ABE$ = ∠ $BAD$, and meets $DC$ produced in $E$, the circles described about △s $BCD$, $BED$ will be equal.

## PROPOSITION 25. PROBLEM.

*An arc of a circle being given, to complete the circle.*



Let $ABC$ be the given arc of a circle:
*it is required to complete the circle.*

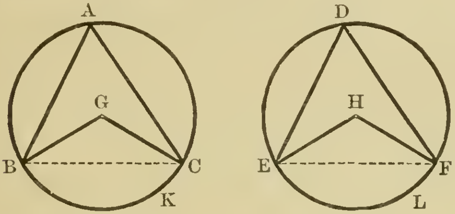
Take any point $B$ in the arc, and join $AB$, $BC$.
Bisect $AB$ and $BC$ at $D$ and $E$; *I.* 10
draw $DF$ and $EF$ respectively $\perp$ $AB$ and $BC$, *I.* 11
and let them meet at $F$.

Because $DF$ bisects the chord $AB$ perpendicularly,
$\therefore$ $DF$ passes through the centre. *III.* 1, *Cor.* 1
Hence also, $EF$ passes through the centre;
$\therefore$ $F$ is the centre.
Hence, with $F$ as centre, and $FA$, $FB$, or $FC$ as radius, the circle may be completed.

1. Prove that $DF$ and $EF$ must meet.
2. Prove the proposition with Euclid's construction, which is: Bisect the chord $AC$ at $D$, draw $DB \perp AC$, meeting the arc at $B$, and join $AB$. At $A$ make $\angle BAE = \angle ABD$, and let $AE$ meet $BD$ or $BD$ produced at $E$. $E$ shall be the centre.
3. Find a point equidistant from three given points. When is the problem impossible?
4. The straight lines bisecting perpendicularly the three sides of a triangle are concurrent.
5. Find a point equidistant from four given points. When is the problem possible?

## PROPOSITION 26. THEOREM.

*In equal circles, or in the same circle, if two angles, whether at the centre or at the circumference, be equal, the arcs on which they stand are equal.*



Let $ABC$, $DEF$ be equal circles, and let $\angle$s $G$ and $H$ at the centres be equal, as also $\angle$s $A$ and $D$ at the $\bigcirc^{ces}$: *it is required to prove that arc $BKC$ = arc $ELF$.*

Join $BC$, $EF$.

Because $\odot$s $ABC$, $DEF$ are equal, *Hyp.*
∴ their radii are equal. *III. Def.* 1, *Cor.* 4

$$\text{In } \triangle\text{s } BGC,\ EHF, \left\{ \begin{array}{l} BG = EH \\ GC = HF \\ \angle G = \angle H; \end{array} \right. \qquad \textit{Hyp.}$$

∴ $BC = EF$. *I.* 4
But because $\angle A = \angle D$,
∴ segment $BAC$ is similar to segment $EDF$; *III. Def.* 13
and they are on equal chords $BC$, $EF$,
∴ segment $BAC$ = segment $EDF$. *III.* 24
Now $\odot ABC = \odot DEF$; *Hyp.*
∴ remaining segment $BKC$ = remaining segment $ELF$;
∴ arc $BKC$ = arc $ELF$.

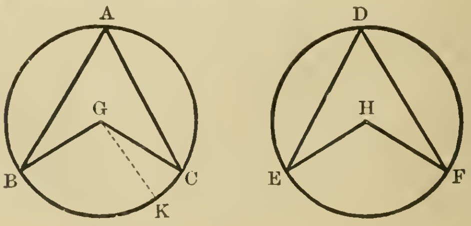
COR.—In equal circles, or in the same circle, those sectors are equal which have equal angles.

1. If $AB$ and $CD$ be two parallel chords in a circle $ACDB$, prove arc $AC =$ arc $BD$, and arc $AD =$ arc $BC$.
2. In equal circles, or in the same circle, if two angles, whether at the centre or at the $\bigcirc^{ce}$ be unequal, that which is the greater stands on the greater arc.
3. If two opposite angles of a quadrilateral inscribed in a circle be equal, the diagonal which does not join their vertices is a diameter of the circle.
4. Any segment of a circle containing a right angle is a semicircle.
5. Any segment of a circle containing an acute angle is greater than a semicircle, and one containing an obtuse angle is less than a semicircle.
6. If two angles at the $\bigcirc^{ce}$ of a circle are supplementary, the sum of the arcs on which they stand = the whole $\bigcirc^{ce}$.
7. Prove the proposition by superposition.
8. If two chords intersect within a circle, the angle they contain is equal to an angle at the centre standing on half the sum of the intercepted arcs.
9. If two chords produced intersect without a circle, the angle they contain is equal to an angle at the centre standing on half the difference of the intercepted arcs.
10. Show how to divide the $\bigcirc^{ce}$ of a circle into 3, 4, 6, 8 equal parts.

---

## PROPOSITION 27. THEOREM.

*In equal circles, or in the same circle, if two arcs be equal, the angles, whether at the centre or at the circumference, which stand on them are equal.*



Let $ABC$, $DEF$ be equal circles, and let arc $BC$ = arc $EF$: *it is required to prove that* $\angle\ BGC = \angle\ EHF$, *and* $\angle\ A = \angle\ D$.

If $\angle\ BGC$ be not = $\angle\ EHF$, one of them must be the greater.

Let $\angle\ BGC$ be the greater, and make $\angle\ BGK = \angle\ EHF$. *I.* 23

Because the circles are equal, and $\angle\ BGK = \angle\ EHF$,

$\therefore$ arc $BK$ = arc $EF$. *III.* 26

But arc $BC$ = arc $EF$; *Hyp.*

$\therefore$ arc $BK$ = arc $BC$, which is impossible.

Hence $\angle\ BGC$ must be = $\angle\ EHF$.

Now, since $\angle\ A$ = half of $\angle\ BGC$, *III.* 20

and $\angle\ D$ = half of $\angle\ EHF$, *III.* 20

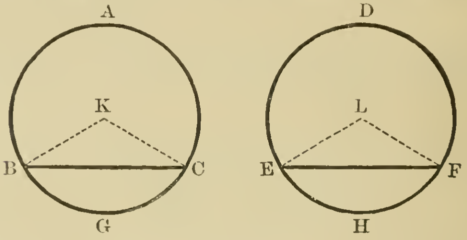
$\therefore \angle\ A = \angle\ D$.

Cor.—In equal circles, or in the same circle, those sectors are equal which have equal arcs.

1. If $AC$ and $BD$ be two equal arcs in a circle $ACDB$, prove chord $AB \parallel$ chord $CD$.
2. In equal circles, or in the same circle, if two arcs be unequal, that angle, whether at the centre or at the ○ce, is the greater which stands on the greater arc.
3. The angle in a semicircle is a right angle.
4. The angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle.
5. If the sum of two arcs of a circle be equal to the whole ○ce, the angles at the ○ce which stand on them are supplementary.
6. Prove the proposition by superposition.
7. Two circles touch each other internally, and a chord of the greater circle is a tangent to the less. Prove that the chord is divided at its point of contact into segments which subtend equal angles at the point of contact of the circles.

## PROPOSITION 28. THEOREM.

*In equal circles, or in the same circle, if two chords be equal, the arcs they cut off are equal, the major arc equal to the major arc, and the minor equal to the minor.*



Let $ABC$, $DEF$ be equal circles, and let chord $BC$ = chord $EF$:

*it is required to prove that major arc* $BAC$ = *major arc* $EDF$, *and minor arc* $BGC$ = *minor arc* $EHF$.

Find $K$ and $L$ the centres of the circles, *III.* 1
and join $BK$, $KC$, $EL$, $LF$.

Because ⊙s $ABC$, $DEF$ are equal, *Hyp.*
∴ their radii are equal. *III. Def.* 1, *Cor.* 4

$$\text{In } \triangle\text{s } BKC,\ ELF, \left\{\begin{array}{l} BK = EL \\ KC = LF \\ BC = EF; \end{array}\right. \qquad \textit{Hyp.}$$

∴ $\angle K = \angle L$; *I.* 8
∴ arc $BGC$ = arc $EHF$. *III.* 26
But $\bigcirc^{ce}\ ABC = \bigcirc^{ce}\ DEF$; *III. Def.* 1, *Cor.* 4
∴ remaining arc $BAC$ = remaining arc $EDF$.

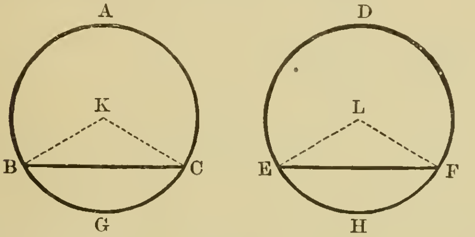
1. If $AC$ and $BD$ be two equal chords in a circle $ACDB$, prove chord $AB$ ∥ chord $CD$.
2. Hence devise a method of drawing through a given point a straight line parallel to a given straight line.

3. If two equal circles cut one another, any straight line drawn through one of the points of intersection will meet the circles again in two points which are equidistant from the other point of intersection.

---

## PROPOSITION 29. THEOREM.

*In equal circles, or in the same circle, if two arcs be equal, the chords which cut them off are equal.*



Let $ABC$, $DEF$ be equal circles, and let arc $BGC$ = arc $EHF$:

*it is required to prove that chord* $BC$ = *chord* $EF$.

Find $K$ and $L$ the centres of the circles, *III.* 1
and join $BK$, $KC$, $EL$, $LF$.

Because the circles are equal, *Hyp.*
∴ their radii are equal. *III. Def.* 1, *Cor.* 4
And because the circles are equal, and arc $BGC$ = arc $EHF$,
∴ $\angle K = \angle L$. *III.* 27

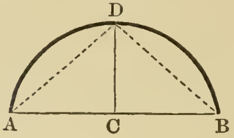
$$\text{In } \triangle\text{s } BKC, ELF, \left\{\begin{array}{l} BK = EL \\ KC = LF \\ \angle K = \angle L; \end{array}\right.$$

∴ $BC = EF$. *I.* 4

1. If $AC$ and $BD$ be two equal arcs in a circle $ACDB$, prove chord $AD$ = chord $BC$.
2. Prove the proposition by superposition.

## PROPOSITION 30. PROBLEM.

*To bisect a given arc.*



Let $ADB$ be the given arc:
*it is required to bisect it.*

Draw the chord $AB$, and bisect it at $C$; *I.* 10
from $C$ draw $CD \perp AB$, and meeting the arc at $D$. *I.* 11

$D$ is the point of bisection.

Join $AD$, $BD$.

$$\text{In } \triangle\text{s } ACD,\ BCD, \left\{\begin{array}{l} AC = BC \qquad \textit{Const.} \\ CD = CD \\ \angle ACD = \angle BCD; \end{array}\right.$$

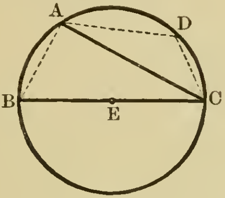
$\therefore AD = BD$. *I.* 4

But in the same circle equal chords cut off equal arcs, the major arc being = the major arc, and the minor = the minor;
and $AD$ and $BD$ are both minor arcs, since $DC$ if produced would be a diameter; *III.* 1, *Cor.* 1
$\therefore$ arc $AD$ = arc $BD$. *III.* 28

1. If two circles cut one another, the straight line joining their centres, being produced, bisects all the four arcs.
2. A diameter of a circle bisects the arcs cut off by all the chords to which it is perpendicular.
3. Bisect the arc $ADB$ without joining $AB$.
4. Prove $\triangle\ DAB$ greater than any other triangle on the same base $AB$, and having its vertex on the arc $ADB$.

## PROPOSITION 31. THEOREM.

*An angle in a semicircle is a right angle; an angle in a segment greater than a semicircle is less than a right angle; and an angle in a segment less than a semicircle is greater than a right angle.*



Let *ABC* be a circle, of which *E* is the centre and *BC* a diameter; and let any chord *AC* be drawn dividing the circle into the segment *ABC* which is greater than a semicircle, and the segment *ADC* which is less than a semicircle:

*it is required to prove*

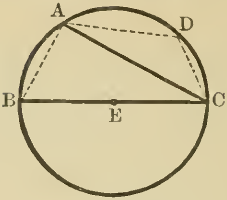
(1) ∠ *in semicircle BAC = a rt.* ∠ ;
(2) ∠ *in segment ABC less than a rt.* ∠ ;
(3) ∠ *in segment ADC greater than a rt.* ∠ .

Join *AB*;
take any point *D* in arc *ADC*, and join *AD*, *CD*.

(1) Because an angle at the $\bigcirc^{ce}$ of a circle is half of the angle at the centre which stands on the same arc; *III.* 20

∴ ∠ *BAC* = half of the straight ∠ *BEC*,
= half of two rt. ∠ s, *III. Def.* 21
= a rt. ∠ .

(2) Because ∠ *BAC* + ∠ *B* is less than two rt. ∠ s, *I.* 17
and ∠ *BAC* = a rt. ∠ ;
∴ ∠ *B* is less than a rt. ∠.



(3) Because $ABCD$ is a quadrilateral inscribed in the circle,

$\therefore \angle B + \angle D =$ two rt. $\angle$s. *III.* 22

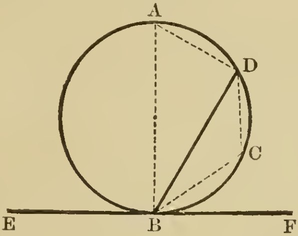
But $\angle B$ is less than a rt. $\angle$;

$\therefore \angle D$ is greater than a rt. $\angle$.

1. Circles described on the equal sides of an isosceles triangle as diameters intersect at the middle point of the base.
2. Circles described on any two sides of a triangle as diameters intersect on the third side or the third side produced.
3. Use the first part of the proposition to solve I. 11, and I. 12.
4. Solve III. 1 by means of a set square.
5. Solve III. 17, Case 3, by the following construction: Join $AE$, and on it as diameter describe a circle cutting the given circle at $B$ and $C$. $B$ and $C$ are the points of contact of the tangents from $A$.
6. If one circle pass through the centre of another, the angle in the exterior segment of the latter circle is acute.
7. If one circle be described on the radius of another circle, any chord in the latter drawn from the point in which the circles meet is bisected by the former.
8. If two circles cut one another, and from one of the points of intersection two diameters be drawn, their extremities and the other point of intersection will be in one straight line.
9. Use the first part of the proposition to find a square equal to the difference of two given squares.
10. The middle point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.
11. State and prove a converse of the preceding deduction.
12. Two circles touch externally at $A$; $B$ and $C$ are points of contact of a common tangent to the two circles. Prove $\angle BAC$ right.

## PROPOSITION 32. THEOREM.

*If a straight line be a tangent to a circle, and from the point of contact a chord be drawn, the angles which the chord makes with the tangent shall be equal to the angles in the alternate segments of the circle.*



Let *ABC* be a circle, *EF* a tangent to it at the point *B*, and from *B* let the chord *BD* be drawn :
*it is required to prove* $\angle$ *DBF* = *the* $\angle$ *in the segment BAD,*
*and* $\angle$ *DBE* = *the* $\angle$ *in the segment BCD.*

From *B* draw $BA \perp EF$; *I.* 11
take any point *C* in the arc *BD*, and join *BC*, *CD*, *DA*.

Because *BA* is drawn $\perp$ the tangent *EF* from the point of contact,

$\therefore$ *BA* passes through the centre of the circle ; *III.* 19
$\therefore$ $\angle ADB$, being in a semicircle, = a rt. $\angle$ ; *III.* 31
$\therefore$ $\angle BAD + \angle ABD$ = a rt. $\angle$, *I.* 32
$= \angle ABF$.

From these equals take away the common $\angle ABD$;
$\therefore$ $\angle BAD = \angle DBF$.

Again, because *ABCD* is a quadrilateral in a circle,
$\therefore$ $\angle A + \angle C = 2$ rt. $\angle$s. *III.* 22
But $\angle DBF + \angle DBE = 2$ rt. $\angle$s ; *I.* 13
$\therefore$ $\angle A + \angle C = \angle DBF + \angle DBE$.

Now $\angle A = \angle DBF$;

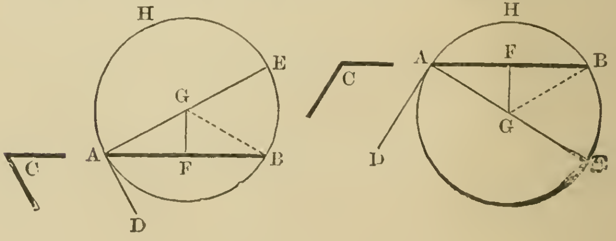
$\therefore \angle C = \angle DBE$.

1. The chord which joins the points of contact of parallel tangents to a circle is a diameter.
2. If two circles touch each other externally or internally, any straight line passing through the point of contact cuts off pairs of similar segments.
3. If two circles touch each other externally or internally, and two straight lines be drawn through the point of contact, the chords joining their extremities are parallel.
4. If two tangents be drawn to a circle from any point, the angle contained by the tangents is double the angle contained by the chord of contact, and the diameter drawn through either point of contact.
5. Enunciate and prove the converse of the proposition.
6. *A* and *B* are two points on the $\bigcirc^{ce}$ of a given circle. With *B* as centre and *BA* as radius describe a circle cutting the given circle at *C* and *AB* produced at *D*. Make arc *DE* = arc *DC*, and join *AE*. *AE* is a tangent to the given circle.
7. Show that this proposition is a particular case either of III. 21, or of III. 22, Cor.

---

## PROPOSITION 33. PROBLEM.

*On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.*



Let $AB$ be the given straight line, $\angle C$ the given angle: *it is required to describe on $AB$ a segment of a circle which shall contain an angle $= \angle C$.*

At $A$ make $\angle BAD = \angle C$. *I.* 23
From $A$ draw $AE \perp AD$; *I.* 11
bisect $AB$ at $F$, *I.* 10
and draw $FG \perp AB$. *I.* 11

Join $BG$.

$$\text{In } \triangle\text{s } AFG, BFG, \left\{\begin{array}{l} AF = BF \qquad \textit{Const.} \\ FG = FG \\ \angle AFG = \angle BFG; \end{array}\right.$$

$\therefore AG = BG$; *I.* 4

$\therefore$ a circle described with centre $G$ and radius $AG$ will pass through $B$.

Let this circle be described, and let it be $AHB$.

The segment $AHB$ is the required segment.

Because $AD$ is $\perp AE$, a diameter of the $\odot AHB$,

$\therefore AD$ is a tangent to the circle. *III.* 16

Because $AB$ is a chord of the circle drawn from the point of contact $A$,

$\therefore$ the angle in the segment $AHB = \angle BAD$, *III.* 32

$= \angle C$.

1. Show that the point $G$ could be found equally well by making at $B$ an angle $= \angle BAE$, instead of bisecting $AB$ perpendicularly.

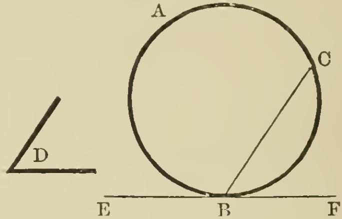
Construct a triangle, having given:

2. The base, the vertical angle, and one side.
3. The base, the vertical angle, and the altitude.
4. The base, the vertical angle, and the perpendicular from one end of the base on the opposite side.
5. The base, the vertical angle, and the sum of the sides.
6. The base, the vertical angle, and the difference of the sides.

[Several other methods of solving this proposition will be found in T. S. Davies's edition (12th) of Hutton's *Course of Mathematics*, vol. i. pp. 389, 390.]

## PROPOSITION 34. PROBLEM.

*From a given circle to cut off a segment which shall contain an angle equal to a given angle.*



Let $ABC$ be the given circle, and $\angle\ D$ the given angle: *it is required to cut off from $\odot\ ABC$ a segment which shall contain an angle* $= \angle\ D$.

Take any point $B$ on the $\bigcirc^{ce}$, and at $B$ draw the tangent $EF$. *III.* 17

At $B$ make $\angle\ FBC = \angle\ D$. *I.* 23

The segment $BAC$ is the required segment.

Because $EF$ is a tangent to the circle, and the chord $BC$ is drawn from the point of contact $B$,

∴ the angle in the segment $BAC = \angle\ FBC$, *III.* 32

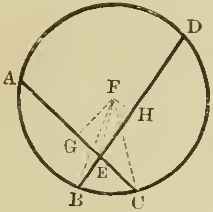
$= \angle\ D$.

Through a given point either within or without a given circle, draw a straight line cutting off a segment containing a given angle. Is the problem always possible?

## PROPOSITION 35. THEOREMS.

*If two chords of a circle cut one another, the rectangle contained by the segments of the one shall be equal to the rectangle contained by the segments of the other.*

*Conversely: If two straight lines cut one another so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, the four extremities of the two straight lines are concyclic.*[*]



(1) Let $AC$, $BD$ two chords of the circle $ABC$ cut one another at $E$:

*it is required to prove* $AE \cdot EC = BE \cdot ED$.

Find $F$ the centre of the $\odot\, ABC$, *III.* 1
and from it draw $FG \perp AC$, and $FH \perp BD$. *I.* 12
Join $FB$, $FC$, $FE$.

Because $FG$ drawn from the centre is $\perp AC$,
$\therefore$ $AC$ is bisected at $G$. *III.* 3

Because $AC$ is divided into two equal segments $AG$, $GC$, and also internally into two unequal segments $AE$, $EC$,

$$\therefore\ AE \cdot EC = GC^2 - GE^2, \qquad \textit{II. } 5$$

$$= (FC^2 - FG^2) - (FE^2 - FG^2), \quad \textit{I. } 47, \textit{ Cor.}$$

$$= FC^2 - FE^2.$$

Similarly, $BE \cdot ED = FB^2 - FE^2$.

But $FC^2 = FB^2$;

$\therefore$ $FC^2 - FE^2 = FB^2 - FE^2$;

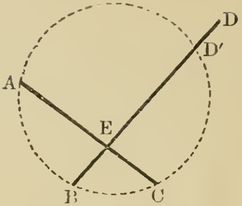
$\therefore$ $AE \cdot EC = BE \cdot ED$.

(2) Let the two straight lines $AC$, $BD$ cut one another at $E$, so that $AE \cdot EC = BE \cdot ED$:

*it is required to prove the four points* $A$, $B$, $C$, $D$ *concyclic.*

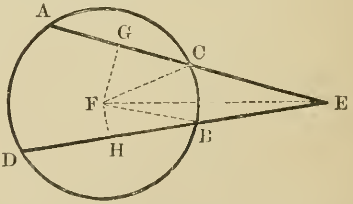
---

[*] The second part of this proposition is not given by Euclid.



Since a circle can always be described through three points which are not in the same straight line,
let a circle be described through $A$, $B$, $C$. *III.* 1, *Cor.* 2
If this circle do not pass also through $D$, let it cut $BD$ or $BD$ produced at the point $D'$;
then $AE \cdot EC = BE \cdot ED'$. *III.* 35
But $AE \cdot EC = BE \cdot ED$; *Hyp.*
$\therefore\ BE \cdot ED' = BE \cdot ED$;
$\therefore\ ED' = ED$, which is impossible;
$\therefore$ the circle which passes through $A$, $B$, $C$ must pass also through $D$.

Cor.—If two chords of a circle when produced cut one another, the rectangle contained by the segments of the one shall be equal to the rectangle contained by the segments of the other; and conversely.



Let $AC$, $BD$, two chords of the $\odot\ ABC$, cut one another when produced at $E$:
*it is required to prove* $AE \cdot EC = BE \cdot ED$.

Find $F$ the centre of the $\odot\, ABC$, *III.* 1
and from it draw $FG \perp AC$, and $FH \perp BD$. *I.* 12
Join $FB$, $FC$, $FE$.

Because $FG$ drawn from the centre is $\perp AC$,
$\therefore$ $AC$ is bisected at $G$. *III.* 3
Because $AC$ is divided into two equal segments $AG$, $GC$, and also externally into two unequal segments $AE$, $EC$,
$\therefore$ $AE \cdot EC = GE^2 - GC^2$; *II.* 6
$= (FE^2 - FG^2) - (FC^2 - FG^2)$, *I.* 47, *Cor.*
$= FE^2 - FC^2$.
Similarly, $BE \cdot ED = FE^2 - FB^2$.
But $FC^2 = FB^2$;
$\therefore$ $FE^2 - FC^2 = FE^2 - FB^2$;
$\therefore$ $AE \cdot EC = BE \cdot ED$.

The converse is proved in exactly the same way as the converse of the proposition.

Note.—It was proved in the proposition that
$AE \cdot EC = FC^2 - FE^2$.
Now, if the $\odot\, ABC$ and the point $E$ be fixed, $FC$ and $FE$ are constant lengths, and $\therefore$ $FC^2 - FE^2$ is a constant magnitude.
Hence $AE \cdot EC$ is constant.
But $AC$ is any chord through $E$;
$\therefore$ the rectangles contained by the segments of all the chords that can be drawn through $E$ are constant;
or, in other words, if a variable chord pass through a fixed point inside a circle, the rectangle contained by the segments which the point makes on it is constant.
This constant value may be called the *internal potency* of the point with respect to the circle.
It was proved in the cor. that $AE \cdot EC = FE^2 - FC^2$.
Hence, as before, if the $\odot\, ABC$ and the point $E$ be fixed, $AE \cdot EC$ is constant;
that is, if a variable chord pass through a fixed point outside a circle, the rectangle contained by the segments which the point makes on it is constant.

This constant value may be called the *external potency* of the point with respect to the circle.

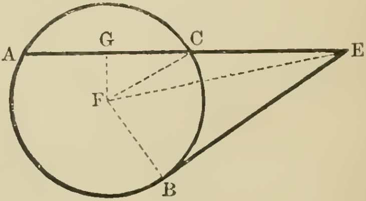
When the point is situated on the ○ᶜᵉ of the circle, its potency with respect to the circle is zero.

[The phrase 'potency of a point with respect to a circle' is due to Steiner. See Jacob Steiner's *Gesammelte Werke*, vol. i. p. 22.]

1. If two circles intersect, and through any point in their common chord two other chords be drawn, one in each circle, their four extremities are concyclic.
2. $ABC$ is a triangle, $AX$, $BY$, $CZ$ the perpendiculars from its vertices on the opposite sides, intersecting at $O$. Prove $AO \cdot OX = BO \cdot OY = CO \cdot OZ$.
3. $ABC$ is a triangle, right-angled at $C$; from any point $D$ in $AB$, or $AB$ produced, a perpendicular to $AB$ is drawn, meeting $AC$, or $AC$ produced, in $E$. Prove $AB \cdot AD = AC \cdot AE$.
4. $ABC$ is any triangle; $D$ and $E$ are two points on $AB$ and $AC$, or on $AB$ and $AC$ produced either through the vertex or below the base, such that $\angle ADE = \angle ACB$. Prove $AB \cdot AD = AC \cdot AE$.
5. Through a point $P$ within a circle a chord $APB$ is drawn such that $AP \cdot PB =$ a given square. Determine the square.
6. Prove VI. B, and VI. C.

## PROPOSITION 36. THEOREM.

*If from a point without a circle a secant and a tangent be drawn to the circle, the rectangle contained by the secant and its external segment shall be equal to the square on the tangent.*



Let $ABC$ be a circle, and from the point $E$ without it let there be drawn a secant $ECA$ and a tangent $EB$: *it is required to prove* $AE \cdot EC = EB^2$.

Find $F$ the centre of the $\odot\ ABC$, *III.* 1
and from it draw $FG \perp AC$. *I.* 12
Join $FB$, $FC$, $FE$.

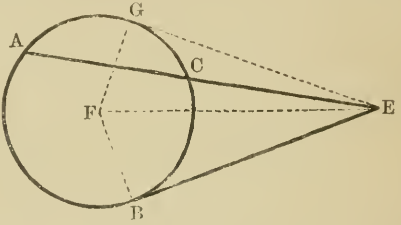
Because $FB$ is drawn from the centre of the circle to $B$, the point of contact of the tangent $EB$,
$\therefore \angle FBE$ is right. *III.* 18
Because $FG$, drawn from the centre, is $\perp AC$,
$\therefore AC$ is bisected at $G$. *III.* 3
Because $AC$ is divided into two equal segments $AG$, $GC$, and also externally into two unequal segments $AE$, $EC$,

$$\begin{aligned} \therefore\quad AE \cdot EC &= GE^2 - GC^2, && \textit{II. } 6\\ &= (FE^2 - FG^2) - (FC^2 - FG^2), && \textit{I. } 47, \textit{ Cor.}\\ &= FE^2 - FC^2,\\ &= FE^2 - FB^2,\\ &= EB^2. && \textit{I. } 47, \textit{ Cor.} \end{aligned}$$

1. Prove the proposition when the secant passes through the centre of the circle. (Euclid gives this particular case.)
2. If two circles intersect, their common chord produced bisects their common tangents.
3. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal.
4. $ABC$ is a triangle, $AX$, $BY$, $CZ$ the perpendiculars from its vertices on the opposite sides. Prove $AC \cdot AY = AB \cdot AZ$, $BC \cdot BX = BA \cdot BZ$, $CA \cdot CY = CB \cdot CX$.
5. From a given point as centre describe a circle to cut a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.
6. Show, by revolving the secant $EBD$ (fig. to III. 35, Cor.) round $E$, that this proposition is a particular case of III. 35, Cor.

## PROPOSITION 37. THEOREM.

*If from a point without a circle two straight lines be drawn, one of which cuts the circle, and the other meets it, and if the rectangle contained by the secant and its external segment be equal to the square on the line which meets the circle, that line shall be a tangent.*



Let $ABC$ be a circle, and from the point $E$ without it let there be drawn a secant $ECA$ and a straight line $EB$ to meet the circle; also, let $AE \cdot EC = EB^2$:
*it is required to prove that $EB$ is a tangent to the $\odot$ $ABC$.*

Draw $EG$ touching the circle at $G$, *III.* 17
and join the centre $F$ to $B$, $G$, and $E$.

Then $\angle FGE =$ a rt. $\angle$. *III.* 18

Now, since $EG$ is a tangent, and $ECA$ a secant,

$\therefore \quad EG^2 = AE \cdot EC,$ *III.* 36

$\qquad = EB^2;$ *Hyp.*

$\therefore \quad EG = EB.$

$$\text{In } \triangle\text{s } EBF,\ EGF, \left\{ \begin{array}{l} EB = EG \\ BF = GF \\ EF = EF; \end{array} \right.$$

$\therefore \angle EBF = \angle EGF,$ *I.* 8

$\qquad =$ a rt. $\angle$;

$\therefore$ $EB$ is a tangent to the $\odot$ $ABC$. *III.* 16

1. Prove the proposition indirectly by supposing $EB$ to meet the circle again at $D$.
2. Prove the proposition indirectly by drawing the tangent $EG$ on the other side of $EF$, and using I. 7.
3. Describe a circle to pass through two given points, and touch a given straight line.
4. Describe a circle to pass through one given point, and touch two given straight lines. Show that to this and the previous problem there are in general two solutions.
5. Describe a circle to touch two given straight lines and a given circle. Show that to this problem there are in general four solutions.
6. Describe a circle to pass through two given points, and touch a given circle. Show that to this problem there are in general two solutions.
7. $AB$ is a straight line, $C$ and $D$ two points on the same side of it; find the point in $AB$ at which the distance $CD$ subtends the greatest angle.

[The third, fourth, fifth, and sixth deductions, along with IV. 4, 5, are cases of the general problem of the Tangencies, a subject on which Apollonius of Perga (about 222 B.C.) composed a treatise, now lost. This problem consists in describing a circle to pass through or touch any three of the following nine data: three points, three straight lines, three circles. It comprises ten cases, which, denoting a point by $P$, a straight line by $L$, and a circle by $C$, may be symbolised thus: $PPP$, $PPL$, $PPC$, $PLL$, $PLC$, $PCC$, $LLL$, $LLC$, $LCC$, $CCC$. An excellent historical account of the solutions given to this problem in its various cases will be found in an article by T. T. Wilkinson, 'De Tactionibus,' in the *Transactions of the Historic Society of Lancashire and Cheshire* (1872). To the authorities there mentioned should be added *Das Problem des Apollonius*, by C. Hellwig (1856); *Das Problem des Pappus von den Berührungen*, by W. Berkhan (1857); 'The Tangencies of Circles and of Spheres,' by Benjamin Alvord, published in 1855 in the 8th vol. of the *Smithsonian Contributions*, and 'The Intersection of Circles and the Intersection of Spheres,' by the same author in the *American Journal of Mathematics*, vol. v., pp. 25–44.]

# APPENDIX III.

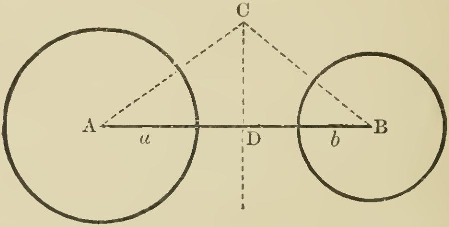
---

## RADICAL AXIS.

DEF. 1.—The locus of a point whose potencies (both external or both internal) with respect to two circles are equal, is called the *radical axis*[*] of the two circles.

### PROPOSITION 1.

*The radical axis of two circles is a straight line perpendicular to the line of centres of the two circles.*



Let $A$ and $B$ be the centres of the given circles, whose radii are $a$ and $b$, and suppose $C$ to be any point on the required locus.

Join $CA$, $CB$, and from $C$ draw $CD \perp AB$ the line of centres.

Since the potency of $C$ with respect to circle $A = AC^2 - a^2$, *Def.*
and since the potency of $C$ with respect to circle $B = BC^2 - b^2$; *Def.*

$\therefore AC^2 - a^2 = BC^2 - b^2$;

$\therefore AC^2 - BC^2 = a^2 - b^2$.

But since the circles $A$ and $B$ are given, their radii ($a$ and $b$) are constant;

$\therefore$ the squares on the radii ($a^2$ and $b^2$) are constant;

$\therefore$ the difference of the squares on the radii ($a^2 - b^2$) is constant;

$\therefore AC^2 - BC^2$ is constant.

Hence the locus of $C$ is a straight line $\perp AB$. *App. II.* 5

---

[*] This name, as well as that of 'radical centre,' was introduced by L. Gaultier de Tours. See *Journal de l'École polytechnique*, 16e cahier, tome ix. (1813), pp. 139, 143.

Cor. 1.—Tangents drawn to the two circles from any point in their radical axis are equal.

Cor. 2.—The radical axis of two circles bisects their common tangents. Hence may be derived a method of drawing the radical axis of two circles.

Cor. 3.—If the two circles are exterior to each other and have no common point, the radical axis is situated outside both circles.

Cor. 4.—If the two circles touch each other either externally or internally, their radical axis consists of the common tangent at the point of contact.

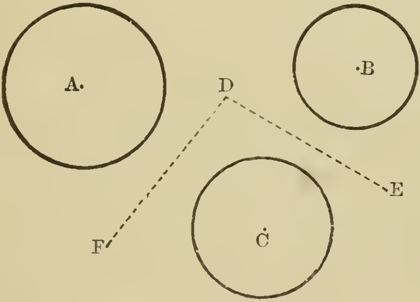
Cor. 5.—If the two circles intersect each other, their radical axis consists of their common chord produced.

Cor. 6.—If one circle is inside the other and does not touch it, their radical axis is situated outside both circles.

Cor. 7.—The radical axis of two unequal circles is nearer to the centre of the small circle than to the centre of the large one, but nearer to the $\bigcirc^{ce}$ of the large circle than to the $\bigcirc^{ce}$ of the small one.

### Proposition 2.

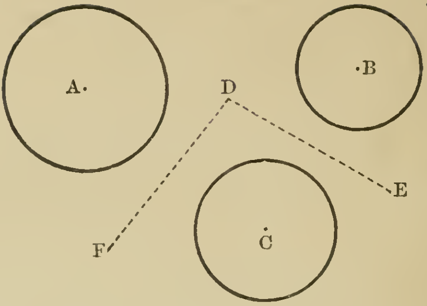
*The radical axes of three circles taken in pairs are concurrent.**



Let *A*. *B*. *C* be three circles, whose radii are *a*, *b*, *c*: *it is required to prove that the radical axis of A and B, that of B and C, and that of C and A all meet at one point.*

---

* This theorem, in one of its cases, is attributed to Monge (1746-1818), in Poncelet's *Propriétés Projectives des Figures*, § 71.



Suppose the centres of the three circles not to be in the same straight line.

Then $DE$, the radical axis of $B$ and $C$, and $DF$, the radical axis of $C$ and $A$, will meet at some point $D$;

for they are respectively ⊥ $BC$ and $CA$, and $BC$ and $CA$ are not in the same straight line.

Since $D$ is a point on the radical axis of $B$ and $C$;

$\therefore\ BD^2 - b^2 = CD^2 - c^2$.

Since $D$ is a point on the radical axis of $C$ and $A$;

$\therefore\ CD^2 - c^2 = AD^2 - a^2$;

$\therefore\ AD^2 - a^2 = BD^2 - b^2$;

$\therefore$ $D$ is a point on the radical axis of $A$ and $B$,

that is, the radical axis of $A$ and $B$ passes through $D$.

Def. 2.—The point of concourse of the radical axes of three circles taken in pairs, is called the *radical centre* of the three circles.

Cor. 1.—When the three circles all cut one another, the radical centre lies either within or without all the three circles.

Cor. 2.—When the centres of the three circles are in one straight line, the radical axes are all parallel, and the radical centre therefore is infinitely distant.

Cor. 3.—When the three circles all touch one another at the same point, the common tangent at that point is the radical axis of all three, and the radical centre therefore is indeterminate—that is, any point on the common radical axis will be a radical centre.

Cor. 4.—In all other cases the radical centre is outside the three circles.

Cor. 5.—If from the radical centre tangents be drawn to the three circles, their points of contact will be concyclic.

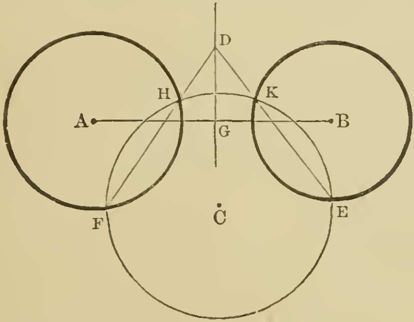
Cor. 6.—If there be several points from which equal tangents can be drawn to three circles, these three circles must have the same radical axis, and the several points must be situated on it.

Cor. 7.—The orthocentre of a triangle is the radical centre of the circles whose diameters are the sides of the triangle, and also the radical centre of the circles whose diameters are the segments of the perpendiculars between the orthocentre and the vertices.

---

## Proposition 3.

*To find the radical axis of two circles which have no common point.*



Let $A$ and $B$ be the two circles.

Describe any third circle $C$ so as to cut the circles $A$ and $B$. Draw $FH$ the common chord of $A$ and $C$, and $EK$ the common chord of $B$ and $C$, and let them meet at $D$.

From $D$ draw $DG \perp AB$.

Then $FD$ is the radical axis of $A$ and $C$, and $ED$ the radical axis of $B$ and $C$; *App. III.* 1, *Cor.* 5

$\therefore$ $D$ is the radical centre of $A$, $B$, and $C$; *App. III.* 2

$\therefore$ $D$ is a point on the radical axis of $A$ and $B$;

$\therefore$ $DG$ is the radical axis of $A$ and $B$. *App. III.* 1

Cor. 1.—The radical axis of $A$ and $B$ may also be obtained thus: After finding $D$, draw a fourth circle to intersect $A$ and $B$. A second pair of common chords will thus be obtained whose intersection will determine another point on the radical axis of $A$ and $B$. Join $D$ with this other point.

Cor. 2.—The radical centre of three circles which have no common point may be found by describing two circles each of which shall cut all the three given circles.

### DEDUCTIONS.

1. Find a point inside a triangle at which the three sides shall subtend equal angles. Is this always possible?
2. Given two intersecting circles, to draw, through one of the points of intersection, a straight line terminated by the circles, and such that (*a*) the sum, (*b*) the difference, of the two chords may = a given length.
3. Of all the straight lines which can be drawn from two given points to meet on the convex $\bigcirc^{ce}$ of a circle, the sum of those two will be the least, which make equal angles with the tangent at the point of concourse.
4. With the extremities of the diameter of a semicircle as centres, any two other semicircles are drawn touching each other externally, and a straight line is drawn to touch them both. Prove that this straight line will also touch the original semicircle.
5. Find a point in the diameter produced of a given circle, such that a tangent drawn from it to the circle shall be of given length.
6. $ABC$ is a triangle having $\angle\ BAC$ acute; prove $BC^2$ less than $AB^2 + AC^2$ by twice the square on the tangent drawn from $A$ to the circle of which $BC$ is a diameter.
7. $ABC$ is a triangle, $AX$, $BY$, $CZ$, the perpendiculars from its vertices on the opposite sides. Prove that these perpendiculars bisect the angles of $\triangle\ XYZ$, and that $\triangle$s $AYZ$, $XBZ$, $XYC$, $ABC$ are mutually equiangular.
8. If the perpendiculars of a triangle be produced to meet the circle circumscribed about the triangle, the segments of these perpendiculars between the orthocentre and the $\bigcirc^{ce}$ are bisected by the sides of the triangle.
9. If $O$ be the orthocentre of $\triangle\ ABC$, the circles circumscribed about $\triangle$s $ABC$, $AOB$, $BOC$, $COA$ are equal.
10. If $D$, $E$, $F$ be situated respectively on $BC$, $CA$, $AB$, the sides of $\triangle\ ABC$, the $\bigcirc^{ces}$ of the circles circumscribed about the three $\triangle$s $AEF$, $BFD$, $CDE$ will pass through the same point.

11. If on the three sides of any triangle equilateral triangles be described outwardly, the straight lines joining the circumscribed centres of these triangles will form an equilateral triangle.

Construct a triangle, having given the base, the vertical angle, and

12. The perpendicular from the vertex to the base.
13. The median to the base.
14. The projection of the vertex on the base.
15. The point where the bisector of the vertical angle meets the base.
16. The sum or difference of the other sides.

17. Construct a triangle, having given its orthocentric triangle.
18. Draw all the common tangents to two circles. Examine the various cases. (One pair are called *direct*, the other pair *transverse*, common tangents.)
19. Of the chords drawn from any point on the $\bigcirc^{ce}$ of a circle to the vertices of an equilateral triangle inscribed in the circle, the greatest = the sum of the other two.
20. If two chords in a circle intersect each other perpendicularly, the sum of the squares on their four segments = the square on the diameter. (This is the 11th of the *Lemmas* ascribed to Archimedes, 287–212 B.C.)
21. A quadrilateral is inscribed in a circle, and its sides form chords of four other circles. Prove that the second points of intersection of these four circles are concyclic.
22. If four circles be described, either all inside or all outside of any quadrilateral, each of them touching three of the sides or the sides produced, their centres will be concyclic.
23. The opposite sides of a quadrilateral inscribed in a circle are produced to meet. Prove that the bisectors of the two angles thus formed are $\perp$ each other.
24. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the straight line joining the points of concourse = the sum of the squares on the two tangents from these points. (A converse of this is given in Matthew Stewart's *Propositiones Geometricæ*, 1763, Book i., Prop. 39.)
25. If a circle be circumscribed about a triangle, and from the ends of the diameter $\perp$ the base, perpendiculars be drawn to the other two sides, these perpendiculars will intercept on the sides segments = half the sum or half the difference of the sides.

26. In the figure to the preceding deduction, find all the angles which are = half the sum or half the difference of the base angles of the triangle.

27. If from any point in the ○ᶜᵉ of the circle circumscribed about a triangle, perpendiculars be drawn to the sides of the triangle, the feet of these perpendiculars are collinear. (This theorem is frequently attributed to Robert Simson, 1687–1768. I have not been able to find it in his works.)

28. If from any point in the ○ᶜᵉ of the circle circumscribed about a triangle, straight lines be drawn, making with the sides, in cyclical order, equal angles, the feet of these straight lines are collinear.

29. If $P$ be any point in the ○ᶜᵉ of the circle circumscribed about △ $ABC$, $X$, $Y$, $Z$, its projections on the sides $BC$, $CA$, $AB$, the circle which passes through the centres of the circles circumscribed about △s $AZY$, $BXZ$, $CYX$ is constant in magnitude.

30. If a straight line cut the three sides of a triangle, and circles be circumscribed about the new triangles thus formed, these circles will all pass through one point; and this point will be concyclic with the vertices of the original triangle. (Steiner's *Gesammelte Werke*, vol. i. p. 223.)

31. If any number of circles intersect a given circle, and pass through two given points, the straight lines joining the intersections of each circle with the given one will all meet in the same point.

32. A series of circles touch a fixed straight line at a fixed point; show that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

33. $ABCD$ is a quadrilateral having $AB = AD$, and $\angle C = \angle B + \angle D$; prove $AC = AB$ or $AD$.

34. From $C$ two tangents $CD$, $CE$ are drawn to a semicircle whose diameter is $AB$; the chords $AE$, $BD$ intersect at $F$. Prove that $CF$ produced is $\perp AB$. (This is the 12th of the *Lemmas* ascribed to Archimedes, and the preceding deduction is assumed in the proof of it.)

35. On the same supposition, prove that if the chords $AD$, $BE$ intersect at $F'$, $F'C$ produced is $\perp AB$.

36. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal; prove that the common chords of each pair pass through this point.

37. Find a point in the $\bigcirc^{ce}$ of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest, or the least possible.
38. From a given point in the $\bigcirc^{ce}$ of a circle draw a chord which shall be bisected by a given chord in the circle.
39. From a point $P$ outside a circle two secants $PAB$, $PDC$ are drawn to the circle $ABCD$; $AC$, $BD$ are joined and intersect at $O$. Prove that $O$ lies on the chord of contact of the tangents drawn from $P$ to the circle. (See Poudra's *Œuvres de Desargues,* tome i., pp. 189–192, 273, 274.)
40. Hence devise a method of drawing tangents to a circle from an external point by means of a ruler only.

## Loci.

Find the locus of the centres of the circles which touch

1. A given straight line at a given point.
2. A given circle at a given point.
3. A given straight line, and have a given radius.
4. A given circle, and have a given radius.
5. Two given straight lines.
6. Two given equal circles.

7. A series of parallel chords are placed in a circle; find the locus of their middle points.
8. A series of equal chords are placed in a circle; find the locus of their middle points.
9. A series of right-angled triangles are described on the same hypotenuse; find the locus of the vertices of the right angles.
10. A variable chord of a given circle passes through a fixed point; find the locus of the middle point of the chord. Examine the cases when the fixed point is inside the circle, outside the circle, and on the $\bigcirc^{ce}$.
11. Find the locus of the vertices of all the triangles which have the same base, and their vertical angles equal to a given angle.
12. Of the $\triangle$ $ABC$, the base $BC$ is given, and the vertical angle $A$; find the locus of the point $D$, such that $BD$ = the sum of the sides $BA$, $AC$.
13. Of the $\triangle$ $ABC$, the base $BC$ is given, and the vertical angle $A$; find the locus of the point $D$, such that $BD$ = the difference of the sides $BA$, $AC$.

14. $AB$ is a fixed chord in a given circle, and from any point $C$ in the arc $ACB$, a perpendicular $CD$ is drawn to $AB$. With $C$ as centre and $CD$ as radius a circle is described, and from $A$ and $B$ tangents are drawn to this circle which meet at $P$; find the locus of $P$.
15. A quadrilateral inscribed in a circle has one side fixed, and the opposite side constant; find the locus of the intersection of the other two sides, and of the intersection of the diagonals.
16. Two circles touch a given straight line at two given points, and also touch one another; find the locus of their point of contact.
17. Find the locus of the points from which tangents drawn to a given circle may be perpendicular to each other.
18. Find the locus of the points from which tangents drawn to a given circle may contain a given angle.
19. Find the locus of the points from which tangents drawn to a given circle may be of a given length.
20. From any point on the ○<sup>ce</sup> of a given circle, secants are drawn such that the rectangle contained by each secant and its exterior segment is constant; find the locus of the ends of the secants.
21. $A$ is a given point and $BC$ a given straight line; any point $P$ is taken on $BC$, and $AP$ is joined. Find the locus of a point $Q$ taken on $AP$ such that $AP \cdot AQ$ is constant.
22. The hypotenuse of a right-angled triangle is given; find the loci of the corners of the squares described outwardly on the sides of the triangle.
23. A variable chord of a given circle passes through a fixed point, and tangents to the circle are drawn at its extremities; prove that the locus of the intersection of the tangents is a straight line. (This straight line is called the *polar* of the given fixed point, and the given fixed point is called the *pole*, with reference to the given circle. See the reference to Desargues on p. 221.)
24. Examine the case when the fixed point is outside the circle.